MICROLOCAL KERNEL OF PSEUDODIFFERENTIAL OPERATORS AT AN HYPERBOLIC FIXED POINT

JEAN-FRANÇOIS BONY, SETSURO FUJIIE, THIERRY RAMOND, AND MAHER ZERZERI

Abstract. We study the microlocal kernel of h-pseudodifferential operators $\mathrm{Op}_h(p) - z$, where z belongs to some neighborhood of size $\mathcal{O}(h)$ of a critical value of its principal symbol $p_0(x, \xi)$. We suppose that this critical value corresponds to a hyperbolic fixed point of the Hamiltonian flow H_{p_0}. First we describe propagation of singularities at such a hyperbolic fixed point, both in the analytic and in the $\mathcal{C}^\infty$ category. In both cases, we show that the null solution is the only element of this microlocal kernel which vanishes on the stable incoming manifold, but for energies z in some discrete set. For energies z out of this set, we build the element of the microlocal kernel with prescribed data on the incoming manifold. We describe completely the operator which associate the value of this null solution on the outgoing manifold to the initial data on the incoming one. In particular it appears to be a semiclassical Fourier integral operator associated to some natural canonical relation.

1. Introduction

This paper is devoted to the study of the microlocal solutions near $(0,0)$ to the equation $(P-z)u = 0$, where $P = \mathrm{Op}_h(p(x,\xi,h))$ is a self-adjoint h-pseudodifferential operator whose principal symbol can be reduced to

$$p_0(x,\xi) = \sum_{j=1}^{d} \frac{\lambda_j}{2}(\xi_j^2 - x_j^2) + \mathcal{O}((x,\xi)^3), \tag{1.1}$$

for some real and positive λ_j's. The energies z are supposed to lie at distance $\mathcal{O}(h)$ of the critical value $p_0(0,0) = 0$.

Of course such a situation occurs for a Schrödinger operator $-h^2\Delta + V$ when the potential V has a non-degenerate local maximum, and the results of this paper might have many applications to quantum theory, allowing precise study of spectral or scattering quantities attached to these Schrödinger operators.

In this setting, the Hamiltonian vector field associated to P has an hyperbolic fixed point at $(0,0)$, and the stable/unstable manifold theorem ensures the existence of a stable incoming manifold Λ_-, and of a stable outgoing manifold Λ_+ in $T^*\mathbb{R}^d$. The manifold Λ_- (resp. Λ_+) can be described as the union of bicharacteristics $t \mapsto \gamma(t)$ such that $\gamma(t) \to (0,0)$ as $t \to +\infty$ (resp. as $t \to -\infty$). It is therefore a very natural question to ask, if the knowledge of a microlocal solution of the equation $Pu = 0$ in Λ_- determines the solution on Λ_+, thus in a whole neighborhood of the fixed point.

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In the analytic, one-dimensional case, this problem has been given a complete answer by B. Helffer and J. Sjöstrand in their study of Harper's operator [15]. Their reduction to a normal form result (on the operator side), has then been used in several works, as for the study of gaps width for Hill's equation by C. März [18] and the third author [21], or the computation of the scattering matrix at barrier tops [22, 10]. There is also a series of work by Y. Colin de Verdière and B. Parisse [5, 6] about the so-called double-well problem where the same ideas are developed in a $\mathcal{C}^\infty$ setting.

Here we address that question in the d-dimensional case, $d > 1$. We want to stress out the fact that the results by N. Hanges, V. Ivrii or R. Melrose, concerning propagation of singularities for operators with multiple characteristics (see e.g. [12]), do not apply here, since we are not in the case where the symbol factorizes as $p = p_1 p_2$, with p_1, p_2 of principal type. Also, we don't think that a Birkhoff normal form reduction on the classical level can be used to obtain the results we give in this paper. In any case, such a reduction would require a non-resonant assumption on the λ_j's, that we don't need here.

First, we prove some kind of propagation of singularity result, both in the analytic and in the $\mathcal{C}^\infty$ category. In these two categories, we show in Theorem 2.1 and Theorem 2.2 below that, roughly speaking, the null solution is the only microlocal solution of the equation $(P - z)u = 0$ defined in a neighborhood of $(0, 0)$, which vanishes on the stable incoming manifold Λ_-. This holds for energies z in any neighborhood of the critical energy 0 of size $\mathcal{O}(h)$, that do not belong to some discrete subset $\Gamma(h)$. If $z \in \Gamma(h)$, then purely outgoing solutions exist - that is solutions which vanish out of Λ_+.

In the analytic case, our discussion is strongly related to the study of the resonances generated by a critical point of the principal symbol of a Schrödinger operator, and we use the same strategy as J. Sjöstrand in [26] (see also [16] and [28]): Our proof relies on energy estimates rather than on a reduction to a normal form.

In the $\mathcal{C}^\infty$ case, our proof rely also on energy estimates, but these are obtained using quite different ideas from recent works by N. Burq and M. Zworski, S.H. Tang and M. Zworski (see [3] and [31]), together with h-pseudodifferential calculus in some suitable class of symbols.

Then we turn to existence results in the $\mathcal{C}^\infty$ case: For energies z away from the discrete set $\Gamma(h)$, we show the existence and give a representation formula for the solution of $(P-z)u = 0$ with given Cauchy data on Λ_-. Our proof relies heavily on ideas from B. Helffer and J. Sjöstrand in [14], devoted to the study of the tunnel effect between non-resonant potential wells. Thanks to this representation formula, we build a microlocal transition operator, which associates the microlocal value of this solution on Λ_+ to the data on Λ_-. We describe completely this operator (see Theorem 2.6 and Theorem 2.8), which turns out to be a h-Fourier Integral Operator associated to the canonical relation $\Lambda_+ \times \Lambda_-$.

The rest of the paper is organized as follows: In Section 2, we describe precisely our geometrical settings, give our assumptions, and state our results. Section 3 and Section 4 are devoted to the proof of Theorem 2.1 and of Theorem 2.2, concerning the propagation of singularities at the hyperbolic fixed point, respectively in the analytic category, and in the $\mathcal{C}^\infty$ category. Then, in Section 5, we address the question of existence of microlocal solutions of the natural Cauchy problem associated to our geometric setting, and we prove Theorem 2.5. In Section 6 we obtain a precise formula for that solution which is given in Theorem 2.6 and 2.8. Eventually, we have recalled in a short Appendix the results from h-pseudodifferential calculus that we use in Section 4.

2. Assumptions and main results

2.1. Microlocal terminology.

Since our results are of microlocal nature, and since we shall constantly use this vocabulary through the paper, we briefly recall from [24] (see also [17] and [7]) the precise meaning of expressions like "$u = 0$ microlocally in Ω". For $u \in \mathcal{S}'(\mathbb{R}^d)$, we denote $\mathcal{T}u$ the Sjöstrand-FBI-Bargmann transform of u given by

$$\mathcal{T}u(z,h) = c_d(h) \int e^{-(z-y)^2/2h} u(y)dy, \tag{2.1}$$

where $c_d(h) = 2^{-d/2}(\pi h)^{-3d/4}$ is a normalization constant. The function $\mathcal{T}u$ is an holomorphic function of $z \in \mathbb{C}^d$, and $\mathcal{T}$ is isometric from $L^2(\mathbb{R}^d)$ to the Sjöstrand space $H_\Phi(\mathbb{C}^d)$, defined by

$$H_\Phi(\mathbb{C}^d) = L^2(e^{-2\Phi(z)/h}dz) \cap \mathcal{H}(\mathbb{C}^d),\ \Phi(z) = \frac{(\operatorname{Im} z)^2}{2}, \tag{2.2}$$

where $\mathcal{H}(\mathbb{C}^d)$ is the space of holomorphic functions on $\mathbb{C}^d$, and $H_\Phi(\mathbb{C}^d)$ is endowed with the norm

$$\|f\|_{H_\Phi} = \Big(\int |f(z,h)|^2 e^{-2\Phi(z)/h} dz \Big)^{1/2}. \tag{2.3}$$

To the transform $\mathcal{T}$, one also associates a canonical map $\kappa_{\mathcal{T}} : T^*(\mathbb{R}^d) \to \mathbb{C}^d$ defined by

$$\kappa_{\mathcal{T}}(x,\xi) = (x - i\xi, \xi). \tag{2.4}$$

We shall say that a family $(u_h)_h \in \mathcal{S}'(\mathbb{R}^d)$ is a tempered semiclassical distribution if there exists $N_0 > 0$ such that $h^{-N_0}u_h$ is bounded in $\mathcal{S}'(\mathbb{R}^d)$. Such a tempered semiclassical distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ is said to be analytically microlocally 0 in Ω, an open subset of $T^*(\mathbb{R}^d)$, when there exists a constant $\varepsilon > 0$ such that,

$$\|\mathcal{T}u\|_{H_\Phi(\Omega')} = \mathcal{O}(e^{-\varepsilon/h}) \text{ as } h \to 0, \tag{2.5}$$

where $\Omega' = \Pi_1\kappa_{\mathcal{T}}(\Omega) = \{x - i\xi, (x,\xi) \in \Omega\}$. The closed subset of $T^*\mathbb{R}^d$ where $u = (u_h)_h$ is not analytically microlocally equal to 0 is called the microsupport of u, and we denote it by $\mathrm{MS}(u)$.

In the $\mathcal{C}^\infty$ category, one says that $u \in \mathcal{S}'(\mathbb{R}^d)$ is microlocally 0 in Ω when $\|\mathcal{T}u\|_{H_\Phi(\Omega')} = \mathcal{O}(h^\infty)$. As a matter of fact, in this $\mathcal{C}^\infty$ setting, we shall use L^2 norms instead of the above H_Φ norm, and it will be more convenient to use another version the FBI transform: We set, for $z = x - i\xi$,

$$\begin{aligned} \mathcal{T}'u(x,\xi,h) &= c_d(h) e^{-\xi^2/2h} \int e^{-(z-y)^2/2h} u(y)dy \\ &= c_d(h) \int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y)dy. \end{aligned} \tag{2.6}$$

Then $\mathcal{T}'u$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^{2d}$, and $u \in \mathcal{S}'(\mathbb{R}^d)$ is microlocally 0 in Ω if and only if $\|\mathcal{T}'u\|_{L^2(\Omega)} = \mathcal{O}(h^\infty)$. The closed set of points where u is not microlocally 0 is called the frequency set of u, and we shall denote it by $\mathrm{FS}(u)$.

2.2. **The geometrical setting.**

We consider, microlocally near $(0,0)\in T^*\mathbb{R}^d$, a h-pseudodifferential operator

$$P=\operatorname{Op}_h(p(x,\xi,h)), \tag{2.7}$$

with symbol $p(x,\xi,h)\in\mathcal{S}_h^0(1)$ (see the Appendix A for notations and a short review of h-pseudodifferential calculus). We assume that p is real valued and

$$p(x,\xi,h)\sim\sum_{j=0}^{\infty}p_j(x,\xi)h^j, \tag{2.8}$$

where the principal symbol satisfies, up to a symplectic change of variables,

$$p_0(x,\xi)=\xi^2-\frac{1}{4}\sum_{j=1}^{d}\lambda_j^2x_j^2+\mathcal{O}((x,\xi)^3), \tag{2.9}$$

in a neighborhood of $(0,0)$ in $T^*\mathbb{R}^d$. Here we have ordered the λ_j such that

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_d. \tag{2.10}$$

Since we work microlocally near $(0,0)$, we will assume that p has compact support.

As usual, we denote by

$$H_p=\frac{\partial p_0}{\partial\xi}\frac{\partial}{\partial x}-\frac{\partial p_0}{\partial x}\frac{\partial}{\partial\xi} \tag{2.11}$$

the Hamiltonian field of $p_0(x,\xi)$. In the (x,ξ) coordinates, the linearized vector field F_p of H_p at $(0,0)$ is

$$F_p=d_{(0,0)}H_p=\begin{pmatrix}0 & 2I\\ \frac{1}{2}L^2 & 0\end{pmatrix}, \tag{2.12}$$

where L is the $d\times d$ matrix defined as $L=\operatorname{diag}(\lambda_1,\dots,\lambda_d)$. Then, the spectrum of F_p is $\sigma(F_p)=\{-\lambda_d,\dots,-\lambda_1,\lambda_1,\dots,\lambda_d\}$. Associated to the hyperbolic fixed point, we have therefore a natural decomposition of $T_{(0,0)}(T^*\mathbb{R}^d)=\mathbb{R}^{2d}$ in a direct sum of two linear subspaces Λ_+^0 and Λ_-^0, of dimension d, associated respectively to the positive and negative eigenvalues of F_p. These spaces $\Lambda_\pm^0$ are given by

$$\Lambda_\pm^0=\{(x,\xi)\in\mathbb{R}^{2d},\ \xi_j=\pm\frac{\lambda_j}{2}x_j,\ j=1,\dots,d\}. \tag{2.13}$$

The stable/unstable manifold theorem gives us the existence of two smooth Lagrangian manifolds Λ_+ and Λ_-, defined in a vicinity Ω of $(0,0)$, which are invariant under the H_p flow, and whose tangent space at (0,0) are precisely Λ_+^0 and Λ_-^0. In particular, we see that these manifolds can be written as

$$\Lambda_\pm=\{(x,\xi)\in T^*\mathbb{R}^d,\ \xi=\nabla\varphi_\pm(x)\}, \tag{2.14}$$

for some smooth functions φ_+ and φ_-, which can be chosen so that

$$\varphi_\pm(x)=\pm\sum_{j=1}^{d}\frac{\lambda_j}{4}x_j^2+\mathcal{O}(x^3). \tag{2.15}$$

Notice that if P were a Schrödinger operator, that is $p(x,\xi)=\xi^2+V(x)$, we would have $\varphi_+(x)=-\varphi_-(x)$.

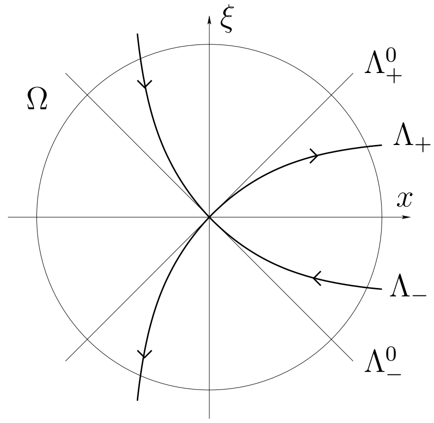

FIGURE 1. The geometry at the singular point.

We shall say that Λ_+ is the outgoing Lagrangian manifold, as Λ_- will be referred to as the incoming Lagrangian manifold associated to the hyperbolic fixed point. Indeed Λ_+ (resp. Λ_-) can be characterized as the set of points $(x, \xi) \in \Omega$ such that $\exp tH_p(x, \xi) \to (0, 0)$ as $t \to -\infty$ (resp. as $t \to +\infty$).

2.3. **Main results.**

Let Ω be a small neighborhood of $(0, 0) \in T^*\mathbb{R}^d$. For $\varepsilon > 0$ small enough, we set $S = \Lambda_- \cap \{(x, \xi);\ |x| = \varepsilon\} \subset \Omega$. For $U \subset \Omega$ a neighborhood of S, we study the microlocal Cauchy problem

$$\begin{cases} (P - z)u = 0 & \text{microlocally in } \Omega, \\ u = u_0 & \text{microlocally in } U. \end{cases} \tag{2.16}$$

Here $u_0 \in L^2(\mathbb{R}^d)$ the microlocal Cauchy data, and we have to suppose that $(P - z)u_0 = 0$ microlocally in S. We assume that Ω is small enough, so that P is of principal type in $\Omega \setminus \{(0, 0)\}$. In particular, we have the usual propagation of singularity results away from the critical point.

First, we address the uniqueness problem for (2.16). If $u_0 = 0$, the solutions have to vanish on the incoming manifold Λ_-, and we ask the question if the corresponding solution is identically 0 in a neighborhood of $(0, 0)$. The first two theorems below state that this is true both in the analytic category and in the $\mathcal{C}^\infty$ category, for complex energies $z \in D(0, C_0 h) = \{z \in \mathbb{C}, |z| < C_0 h\}$, where $C_0 > 0$ is any positive constant, but for z in some discrete set. The existence of this exceptional set should not be too surprising, at least in the analytic case: It corresponds to that of resonances generated by the barrier top, i.e. the existence of "purely outgoing solutions". In the $\mathcal{C}^\infty$ case also, one could have conjectured such a result. Indeed, the principal symbol p_0 can be written in suitable coordinates (y, η) as

$$p_0(x, \xi) = B(y, \eta) y \cdot \eta, \tag{2.17}$$

where B is a smooth map from a neighborhood of $(0, 0)$ in $T^*\mathbb{R}^d$ to the space $\mathcal{M}_d(\mathbb{R})$ of $d \times d$ matrices. Therefore in the one-dimensional case, p_0 factorizes as $p_0 = q_1 q_2$, with q_1 and q_2 of principal type, and using a reduction to a normal form as in the work [12] by N.

Hanges, concerning propagation of singularities for operators with multiple characteristics, this uniqueness result can be shown to hold for z away from the set

$$\{-ih\lambda_1(\alpha+\frac{1}{2})\ ;\ \alpha\in\mathbb{N}\}. \tag{2.18}$$

In the present multidimensional setting, we find it convenient to work with the form (2.17) for our operator, and using h-pseudodifferential calculus in some suitable class of symbols as well as ideas from Sjöstrand in [27] and Burq and Zworski in [3], we show the following result.

Theorem 2.1. *Let Ω be a small neighborhood of $(0,0)\in T^*\mathbb{R}^d$, and $S=\Lambda_-\cap\{(x,\xi);\ |x|=\varepsilon\}\subset\Omega$ for some $\varepsilon>0$ small enough. Assume (2.7)–(2.10). Let N, $C_0>0$ be constants, and $U\subset\Omega$ a neighborhood of S. There exists a neighborhood V of $(0,0)$ such that, for all $z\in D(0,C_0h)\subset\mathbb{C}$, and $u\in L^2(\mathbb{R}^d)$, defined for h small enough with $\|u\|_{L^2}\leq 1$, if*

$$\begin{cases}(P-z)u=0 & \textit{microlocally in } \Omega,\\ u=0 & \textit{microlocally in } U,\end{cases} \tag{2.19}$$

with $d(z,\Gamma(h))>h^N$, then $u=0$ microlocally in V.

Here, $\Gamma(h)$ is a discrete set, defined for any h small enough, such that $\#\Gamma(h)\cap D(0,C_0h)$ is bounded uniformly with respect to h, and $\Gamma(h)\subset\{\operatorname{Im} z<-\delta_0 h\}$ for some $\delta_0>0$.

In the analytic category, we can be as precise about the exceptional set as in the one-dimensional case, changing of course the notion of C^∞-microsupport to that of analytic microsupport. Indeed, if we denote by $\Gamma_0(h)$ the discrete subset of $\mathbb{C}$ defined by

$$\Gamma_0(h)=\Big\{-ih\sum_{j=1}^{d}\lambda_j(\alpha_j+\frac{1}{2}),\ \alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d\Big\}, \tag{2.20}$$

we have the following theorem which is, in some sense, a semiclassical version of a part of the work of Sjöstrand [25].

Theorem 2.2. *Suppose that, in addition to assumptions (2.7)–(2.10), the function $p(x,\xi,h)$ extends holomorphically in a complex neighborhood of $(0,0)$ in $\mathbb{C}^{2d}$. Let ν, $C_0>0$ be constants, and $U\subset\Omega$ a neighborhood of S.*

There exists a neighborhood V of $(0,0)$ such that, for all $z\in D(0,C_0h)\subset\mathbb{C}$, and $u\in L^2(\mathbb{R}^d)$, defined for h small enough with $\|u\|_{L^2}\leq 1$, if

$$\begin{cases}(P-z)u=0 & \textit{analytically microlocally in } \Omega,\\ u=0 & \textit{analytically microlocally in } U,\end{cases} \tag{2.21}$$

with $d(z(h),\Gamma_0(h))>\nu h$, then $u=0$ analytically microlocally in V.

Notice that, as in [26], and using the ideas there, the last assumption in Theorem 2.2 about the distance to the exceptional set can certainly be replaced by a weaker one as in Theorem 2.1, provided the set $\Gamma_0(h)$ is replaced by $\widetilde{\Gamma}_0(h)=\{\lambda_\alpha(h);\ \alpha\in\mathbb{N}^d\}$, where the $\lambda_\alpha(h)$ have an expansion in fractional powers of h and satisfy $\lambda_\alpha(h)=-ih\sum_{1\leq j\leq d}\lambda_j(\alpha_j+1/2)+o(h)$.

Remark 2.3. *In the $\mathcal{C}^\infty$ category, and when the λ_j are $\mathbb{N}$-independent, one can perform WKB construction of purely outgoing solutions for energies $z\in D(0,C_0h)$ such that $d(z,\Gamma_0(h))>\nu h$ (see e.g. [13]). Therefore, in that particular case at least, we have $\Gamma_0(h)\subset\Gamma(h)$.*

Remark 2.4. *The two previous theorems can be proved under slightly more general assumptions. Indeed for Theorem 2.1, it is sufficient to suppose that $P = \mathrm{Op}_h(p)$, where*

- $p(x,\xi;h) = p_0(x,\xi) + hp_1(x,\xi) + h^{1+\varepsilon}p_2(x,\xi;h)$ *for some* $\varepsilon > 0$.
- $p_0(x,\xi)$ *is a real valued* C^∞ *function which can be written, up to a symplectic change of variables,*

$$p_0(x,\xi) = \xi^2 - \frac{1}{4}\sum_{j=1}^{d} \lambda_j^2 x_j^2 + \mathcal{O}((x,\xi)^3),$$

- p_1, $p_2 \in \mathcal{S}_h^0(1)$ *(see Appendix A for the definition of* $\mathcal{S}_h^0(1)$*).*

In that case, the statement $\Gamma(h) \subset \{\operatorname{Im} z < -\delta_0 h\}$ *in Theorem 2.1 should be replaced by* $\Gamma(h) \subset \{\operatorname{Im} z < h(\operatorname{Im} p_1(0,0) - \delta_0)\}$.

For the proof of Theorem 2.2, we have to suppose in addition that p extends as a holomorphic function to a (fixed) neighborhood of $(0,0)$ in $\mathbb{C}^{2d}$.

Now, using ideas from B. Helffer and J. Sjöstrand in [14], we address the question of the existence of solutions for the problem (2.16). As in that paper, to perform our construction we have to suppose that the data is not microlocally supported on some manifold of codimension 1 in Λ_-. Indeed, we know from [14] that there exist functions $\gamma_j^\pm(t,x,\xi)$, polynomials with respect to t, such that, in the precise sense of Definition 5.1 below,

$$\exp(\mp tH_p)(x,\xi) \sim \sum_{j\geq 1} \gamma_j^\pm(t,x,\xi)e^{-\mu_j t}, \quad t \to +\infty, \tag{2.22}$$

for all $(x,\xi) \in \Lambda_\pm$ respectively. Here $(\mu_j)_{j\geq 0}$ is the increasing sequence of linear combinations over $\mathbb{N}$ of the λ_j's. Moreover, the function $\gamma_1^\pm$ is a constant vector with respect to t in $\mathrm{Ker}(d_{(0,0)}H_p \mp \lambda_1)$. We shall also consider x-space projections of the trajectories, and for $\rho \in \Lambda_\pm$ respectively, we shall denote

$$g_1^\pm(\rho) = \Pi_x \gamma_1^\pm(\rho). \tag{2.23}$$

Let m be the number of λ_j's equal to λ_1. We denote by $\widetilde{\Lambda}_\pm$ the subset of $\Lambda_\pm$ which consists in points (x,ξ) such that $\gamma_1^\pm(x,\xi) = 0$. Notice that, using the stable manifold master theorem [1, Theorem 7.2.8], one can see that $\widetilde{\Lambda_\pm}$ is a $\mathcal{C}^\infty$ submanifold of $\Lambda_\pm$ of dimension $d-m$, which is stable under the Hamiltonian flow. As above, we denote by $S \subset \Omega$ the lift in Λ_- of the sphere $\{x \in \mathbb{R}^d;\ |x| = \varepsilon\}$, with $\varepsilon > 0$ small enough.

Theorem 2.5. *Suppose assumptions (2.7)–(2.10) hold. Let C_0, C_1, $\nu > 0$ be constants, $z \in [-C_0h, C_0h] + i[-C_1h, C_1h]$ with $d(z,\Gamma_0(h)) > \nu h$, and $u_0 \in L^2(\mathbb{R}^d)$ be such that $\|u_0\|_{L^2} \leq 1$ with $u_0 = 0$ microlocally in $S \cap \widetilde{\Lambda_-}$ and $(P-z)u_0 = 0$ microlocally in S, then the problem*

$$\begin{cases} (P-z)u = 0 & \text{microlocally in } \Omega, \\ u = u_0 & \text{microlocally in } S, \end{cases} \tag{2.24}$$

has a solution $u(x,z,h)$ such that

$$\|u\|_{L^2} \lesssim h^{-\mathbb{E}\left(\frac{C_1}{\lambda_1} - \frac{\sum \lambda_j}{2\lambda_1} + \frac{d}{2}\right)-1}, \tag{2.25}$$

where $\mathbb{E}(r)$ is the integer part of $r \in \mathbb{R}$.

Moreover, if u_0 is analytic with respect to $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h] \setminus (\Gamma_0(h) + D(0, \nu h))$, then u is also analytic.

We denote by $\mathcal{J}(z)u_0$ the solution of the problem (2.24), which is unique thanks to Theorem 2.1. Using a microlocal partition of unity, we can assume that the initial data u_0 is microlocally supported only in a vicinity of a point $\rho_- = (x^-, \xi^-) \in S \setminus \widetilde{\Lambda_-}$. As for Theorem 2.5, we are unable to calculate the solution $\mathcal{J}(z)u_0$ near every point of Λ_+ and we must avoid some particular set of points $\widetilde{\Lambda_+}(\rho_-)$: Let φ_1 be the solution of the Cauchy problem

$$\begin{cases} (\nabla_\xi p_0(x, \nabla\varphi_+) \cdot \nabla - \lambda_1)\varphi_1 = 0, \\ \nabla\varphi_1(0) = -\lambda_1 g_1^-(\rho_-). \end{cases} \tag{2.26}$$

We set $\widetilde{\Lambda_+}(\rho_-) = \{(x, \xi) \in \Lambda_+;\ \varphi_1(x) = 0\}$. Then, $\widetilde{\Lambda_+}(\rho_-)$ is a $\mathcal{C}^\infty$ submanifold of Λ_+, of codimension 1, which is stable under the Hamiltonian flow, and we can compute $\mathcal{J}(z)u_0$ near any point $\rho_+ = (x^+, \xi^+) \in \Lambda_+ \setminus \widetilde{\Lambda_+}(\rho_-)$. As the operator P is of principal type in a neighborhood of ρ_-, and since u_0 is in the kernel of $P - z$, u_0 is completely determined by its trace on any hypersurface transversal to the flow. Up to a change of variables, we can assume that $x_1 = x_1(\rho_-) = \varepsilon$ is such an hypersurface (taking the first coordinate function to be collinear to $g_1^-(\rho_-)$), and we state the following result in that setting. Eventually, because of (2.9), and for $x' = o(x_1)$, $\xi' = o(x_1)$, the equation $p_0(x, \xi_1, \xi') = 0$ has two solutions

$$\xi_1 = f_\pm(x, \xi') = \pm\frac{\lambda_1}{2}x_1 + o(x_1). \tag{2.27}$$

In the Schrödinger case where $p(x,\xi) = \xi^2 + V(x)$, we would have $f_\pm(x, \xi') = \pm\sqrt{-\xi'^2 - V(x)}$. Then, with these notations, we have the following description for $\mathcal{J}(z)u_0$ near Λ_+.

Theorem 2.6. *We suppose that the assumptions of Theorem 2.5 hold, and that u_0 is microlocally supported only in a vicinity of $\rho_- \in \mathcal{S} \setminus \widetilde{\Lambda_-}$. We set*

$$S(z/h) = \sum_{k=1}^{d} \frac{\lambda_k}{2} - i\frac{z}{h},$$

and we denote by $(\widehat{\mu}_j)_{j\geq 0}$ the increasing sequence of the linear combinations over $\mathbb{N}$ of the $(\mu_k - \mu_1)$. Then, there exists a symbol $d(x, y', z, h) \sim \sum_{j\geq 0} d_j(x, y', z, \ln h) h^{\widehat{\mu}_j/\lambda_1} \in \mathcal{S}_h^0(1)$, with $d_j(x, y', z, \ln h)$ polynomial with respect to $\ln h$, such that

$$\mathcal{J}(z)u_0(x, h) = \frac{h^{S(z/h)/\lambda_1}}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} d(x, y', z, h) e^{i(\varphi_+(x) - \varphi_-(\varepsilon, y'))/h} u_0(\varepsilon, y') dy', \tag{2.28}$$

microlocally near $\rho_+ \in \Lambda_+ \setminus \widetilde{\Lambda_+}(\rho_-)$. The symbols d and d_j are analytic for $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h] \setminus (\Gamma_0(h) + D(0, \nu h))$. Moreover the principal symbol d_0 of d is independent of $\ln h$, and can be written as

$$\begin{aligned} d_0(x, y', z) = & \sqrt{\lambda_1} e^{-id\pi/4} \Gamma\left(S(z/h)/\lambda_1\right) \left(i\lambda_1 \langle g_1^-(\rho^-_{(\varepsilon,y')}) | g_1^+(\rho^+_x)\rangle\right)^{-S(z/h)/\lambda_1} \\ & \left|g_1^-(\rho^-_{(\varepsilon,y')})\right| \left|\det \nabla^2_{y'y'}\varphi_-(\varepsilon, y')\right|^{1/2} \sqrt{\nabla_{\xi_1} p_0(\varepsilon, y', f_-(\varepsilon, y', \eta'), \eta')} \\ & e^{\int_0^{-\infty} (\Delta\varphi_+(x(t)) - \sum \lambda_k/2) dt} \lim_{t\to+\infty} \frac{e^{(\sum \lambda_k/2 - \lambda_1)t}}{\sqrt{\det \frac{\partial y(t, w', y')}{\partial(t, w')}}}, \end{aligned} \tag{2.29}$$

where $\rho_x^\pm = (x, \nabla_x \varphi_\pm(x))$, *and* $x(t)$ *(resp.* $y(t, w', y')$*) denotes the* x*–space coordinate of the hamiltonian curve* $\exp(tH_p)(\rho_x^+)$ *(resp.* $\exp(tH_p)(\varepsilon, , \nabla_y \varphi_-(\varepsilon, y'))$*).*

Remark 2.7. *Notice that, since every quantity in the previous theorem depends smoothly on* ε*, one can also consider the operator* $\mathcal{J}$ *as an operator on* $L^2(\mathbb{R}^d)$*. Indeed one has*

$$\mathcal{J}(z)u_0(x,h) = \frac{h^{S(z/h)/\lambda_1}}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} \widetilde{d}(x,y,z,h) e^{i(\varphi_+(x)-\varphi_-(y))/h} u_0(y) dy. \tag{2.30}$$

where $\widetilde{d}(x,y,z,h) = \chi(y_1) d(x, y_1, y', z, h)$ *for any function* $\chi \in \mathcal{C}_0^\infty(]0, \varepsilon_0[)$*, with* $\varepsilon_0 > 0$ *small enough, such that* $\int \chi(y_1) dy_1 = 1$*. Here* $d(x, y_1, y', z, h)$ *is the symbol given by Theorem 2.6 with* $\varepsilon = y_1$*.*

In order to make even clearer the fact that the microlocal transition operator $\mathcal{J}$ does not really depend on the choice of ε, we shall use the terminology of [29]: For $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$, we denote by $\mathcal{K}_{\rho_\pm}(z)$ the set of distributions u microlocally defined near $\rho_\pm$, such that $(P-z)u = 0$ microlocally near $\rho_\pm$. Notice that, since P is of principal type away from $(0,0)$, there exist $U_\pm$, $V_\pm$ two neighborhoods of $\rho_\pm$ and $(0,0)$ respectively and an elliptic microlocal h-Fourier integral operator (an h-FIO from now on), $\mathcal{U}_\pm(z)$ with canonical transformation $\kappa_\pm : U_\pm \to V_\pm$ such that $\kappa_\pm(\rho_\pm) = (0,0)$ and

$$\mathcal{U}_\pm (P - z) = hD_{x_1} \mathcal{U}_\pm \quad \text{microlocally in } U_\pm. \tag{2.31}$$

Moreover, we have $\kappa_\pm^* \xi_1 := \xi_1 \circ \kappa_\pm(x,\xi) = p(x,\xi)$. (see e.g. [29, Proposition 3.5] in this semiclassical setting). Then $\mathcal{K}_{\rho_\pm}(z)$ can be identified with $\mathcal{D}'(\mathbb{R}^{d-1})$ using $\mathcal{U}_\pm$.

Let $v_- \in \mathcal{D}'(\mathbb{R}^{d-1})$ be microlocally supported in a compact subset of V_-. If u_- is the corresponding element in $\mathcal{K}_{\rho_-}(z)$ and u the solution of (2.24) with initial data u_-, we denote by $\mathcal{I}(z)v_-$ the element of $\mathcal{D}'(\mathbb{R}^{d-1})$ corresponding to u near ρ_+. In other words, we have set

$$\mathcal{I}(z) = \imath^* \, \mathcal{U}_+ \, \mathcal{J}(z) \, \mathcal{U}_-^{-1} \, \pi^*, \tag{2.32}$$

where $\imath : x' \mapsto (0, x')$ and $\pi : (x_1, x') \mapsto x'$.

Theorem 2.8. *Assume* $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$ *and* $d(z, \Gamma_0(h)) > \nu h$*. Then the operator* $\mathcal{I}(z)$ *is a* h*-Fourier integral operator of order* $h^{\operatorname{Re} S(z/h) - \frac{1}{2}}$ *on* $L^2(\mathbb{R}^{d-1})$*, microlocally defined near* $(0,0)$*, analytic with respect to* z*, associated to the canonical relation*

$$\mathcal{C}_{\mathcal{I}} = \Pi \circ \kappa_+(\Lambda_+) \times \Pi \circ \kappa_-(\Lambda_-), \tag{2.33}$$

where $\Pi : (x_1, x', \xi_1, \xi') \mapsto (x', \xi')$*.*

Remark 2.9. *The canonical relation does not depend on the choice of* $\kappa_\pm$ *in the following sense. Suppose that* $\widetilde{U}_\pm$ *are others FIO's, with canonical relation* $\widetilde{\kappa}_\pm$*, as in the discussion before Theorem 2.8. The operators* $\widehat{\mathcal{U}}_\pm = \widetilde{\mathcal{U}}_\pm \mathcal{U}_\pm^{-1}$ *are FIO's with canonical relation* $\widehat{\kappa}_\pm = \widetilde{\kappa}_\pm \circ \kappa_\pm^{-1}$*. Then, we see that* $\widehat{\kappa}_\pm$ *must be of the form*

$$\widehat{\kappa}_\pm(x,\xi) = \big(f_1^\pm(x,\xi), g_{x'}^\pm(x',\xi') + \xi_1 f_2^\pm(x,\xi), \xi_1, g_{\xi'}^\pm(x',\xi') + \xi_1 f_3^\pm(x,\xi)\big), \tag{2.34}$$

where $(x,\xi) = (x^1, x', \xi^1, \xi')$*. Then, Lemma 3.4 of* [29] *implies that* $\imath^* \, \widehat{\mathcal{U}}_\pm \, \pi^*$ *is an FIO on* $L^2(\mathbb{R}^{d-1})$ *with canonical transformation*

$$g_\pm : (x', \xi') \mapsto (g_{x'}^\pm(x',\xi'), g_{\xi'}^\pm(x',\xi')). \tag{2.35}$$

Therefore, if we denote by $\widetilde{\mathcal{I}}(z)$ the same operator as $\mathcal{I}(z)$ but defined through $\widetilde{\mathcal{U}}_{\pm}$ instead of $\mathcal{U}_{\pm}$, we have

$$\mathcal{C}_{\widetilde{\mathcal{I}}}(z) = (g_{+} \times g_{-})(\mathcal{C}_{\mathcal{I}}(z)). \tag{2.36}$$

3. Uniqueness in the analytic case

We prove Theorem 2.2. Since this uniqueness statement is essentially equivalent to the fact that there is no purely outgoing solution, it should not be surprising that our discussion is strongly related to the study of the resonances generated by a maximum of $V(x)$, and we use the same strategy as J. Sjöstrand in [26] (see also [16]), as well of some lemmas from that paper or from [11].

In this section, as for example in Figure 2, we use the same notations for subsets of $T^*\mathbb{R}^d$ and their image in $\mathbb{C}^d$ by $(x, \xi) \mapsto x - i\xi$. We recall that, using also this convention, we shall say that u is microlocally 0 in Ω if $\mathrm{MS}(u) \cap \Omega = \emptyset$,

We work under the assumptions (2.7)–(2.10): We set $P = \mathrm{Op}_h(p)$, where p is a holomorphic function, depending on $h \in]0, 1]$ say, in a (fixed) complex neighborhood of $(0, 0)$ in $\mathbb{C}^{2d}$. We also assume that, up to a linear change of variables, p_0 can be written as

$$p_0(x, \xi) = \sum_{j=1}^{d} \frac{\lambda_j}{2}(\xi_j^2 - x_j^2) + \mathcal{O}((x, \xi)^3) \tag{3.1}$$

for some real and positive λ_j's. We start with this expression for p_0.

As in the discussion of Section 2.2, we work in some neighborhood Ω of the fixed point $(0, 0)$, and we choose $\Omega_1 \Subset \Omega_0 = \Omega$ as in Figure 2. We write

$$\Omega_0 \setminus \Omega_1 = A_{+} \cup A_0 \cup A_{-}, \tag{3.2}$$

where $A_{\pm}$ is close to $\Lambda_{\pm}$. We assume that A_0 is geometrically controlled by A_{-}, that is any point $(x, \xi) \in A_0$ can be written as $\exp tH_p(x_{-}, \xi_{-})$ for some $(x_{-}, \xi_{-}) \in A_{-}$ and some $t > 0$. It is clear that one can find such a configuration when $H_p = F_p$, and Hartmann's Theorem (see e.g. [20]) ensures that we can do so in the general case as well.

We consider the operator on $H_{\Phi}(\Omega)$ defined by

$$\widetilde{P} = \mathcal{T} P \mathcal{T}^*, \tag{3.3}$$

where $\mathcal{T}$ is the FBI transform given in (2.1), and $H_{\Phi}(\Omega)$ is defined in (2.2). Then $\widetilde{P}$ is a pseudodifferential operator in the complex domain (see J. Sjöstrand [24]). Its principal symbol is

$$\widetilde{p}_0(x, \xi) = p_0 \circ \kappa_{\mathcal{T}}^{-1}(x, \xi) = \sum_{j=1}^{d} \frac{\lambda_j}{2}(2\xi_j^2 - 2ix_j\xi_j - x_j^2) + \mathcal{O}((x, \xi)^3). \tag{3.4}$$

First, $u = 0$ microlocally in $\Lambda_{-} \setminus \{(0, 0)\}$, so that we can assume that $u = 0$ microlocally in A_{-} provided Ω_0 is small enough. Since A is geometrically controlled by A_{-}, we get from standard results on propagation of singularities, that, for some $\delta > 0$,

$$\|\mathcal{T} u\|_{H_{\Phi}(A_{-} \cup A_0)} = \mathcal{O}(e^{-\delta/h}). \tag{3.5}$$

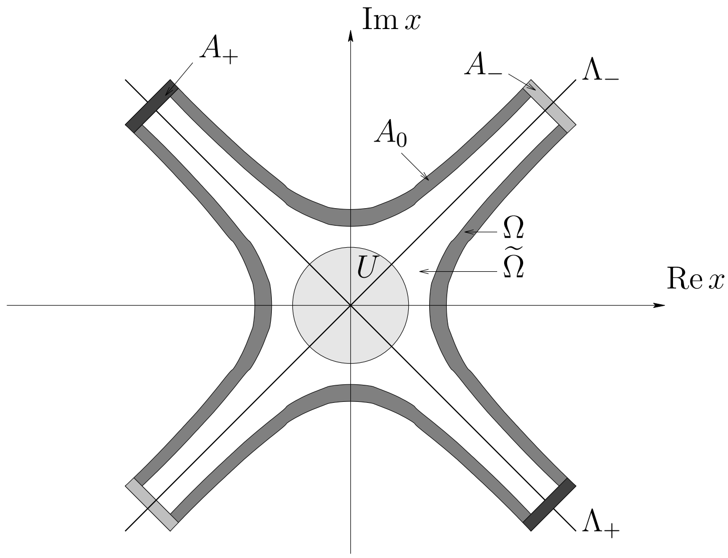

FIGURE 2. The domains.

Now, we choose U an elliptic FIO with complex phase given by

$$\varphi_U(x,y) = \frac{x^2}{2} - xy + \frac{(1-i)}{4}y^2. \tag{3.6}$$

This operator is associated to the complex canonical transform

$$\kappa_U : (x,\xi) \mapsto \left(\frac{2\xi + (1-i)x}{2}, \frac{2\xi - (1+i)x}{2} \right). \tag{3.7}$$

Notice that the operator U cannot be realized on H_Φ since the function $y \mapsto -\operatorname{Im}(\varphi_U(x,y)) + \Phi(y)$ has no saddle point. However, if we set

$$G(x) = -\operatorname{Re} x \operatorname{Im} x, \tag{3.8}$$

then, for $t > 0$ fixed, U is well-defined as an operator from $H_{\Phi+tG}(\Omega_2)$ to $H_{\Psi_t}(\kappa_U(\Omega_3))$, where Ψ_t is some plurisubharmonic function. Here $\Omega_3 \subset \Omega_2$ are suitable neighborhoods of $(0,0)$ depending on t, since the saddle point of $y \mapsto -\operatorname{Im}(\varphi_U(x,y)) + \Phi(y) + tG(y)$ does. From [24], we can invert U by an FIO V from H_{Ψ_t} to $H_{\Phi+tG}$ up to exponentially small errors, taking care of domains. Now we set, after shrinking Ω_2 and Ω_3,

$$Q = U\widetilde{P}V : H_{\Psi_t}(\kappa_U(\Omega_2)) \to H_{\Psi_t}(\kappa_U(\Omega_3)), \tag{3.9}$$

which is a pseudodifferential operator with principal symbol

$$q_0(x,\xi) = \widetilde{p}_0 \circ \kappa_U^{-1}(x,\xi) = \sum_{j=1}^{d} \lambda_j x_j \xi_j + \mathcal{O}((x,\xi)^3). \tag{3.10}$$

Let us recall Proposition 4.4 from [11]:

Proposition 3.1 ([11], Proposition 4.4). *Let $\chi \in C_0^\infty(\kappa_U(\Omega_3))$. There exists a classical symbol $\widetilde{q}(x,h)$ of order 0 such that*

$$\langle \chi Q u, v\rangle_{H_{\Psi_t}(\kappa_U(\Omega_3))} = \langle \widetilde{q} u, v\rangle_{H_{\Psi_t}(\kappa_U(\Omega_3))} + r(u,v), \tag{3.11}$$

where

$$r(u,v)=\mathcal{O}(h^\infty)\|u\|_{H_{\Psi_t}(\kappa_U(\Omega_2))}\|v\|_{H_{\Psi_t}(\kappa_U(\Omega_2))},\tag{3.12}$$

and

$$\widetilde{q}_0(x)=\chi(x)q_0\Big(x,\frac{2}{i}\partial_x\Psi_t(x)\Big).\tag{3.13}$$

We use Proposition 3.1 with $\chi=1$ near $\kappa_U(\Omega_4)$, for some $\Omega_4\Subset\Omega_3$. First, for $x\in\kappa_U(\Omega_3)$, we have

$$\begin{aligned} q_0\Big(x,\frac{2}{i}\partial_x\Psi_t(x)\Big) &= \widetilde{p}_0\Big(y,\frac{2}{i}\partial_y(\Phi+tG)\Big)_{|_{y=\kappa_U^{-1}(x)}}\\ &= p_0\Big(a+2t\partial_z G(a-ib), b-2it\partial_z G(a-ib)\Big)_{|_{z=a-ib=\kappa_U^{-1}(x)}},\end{aligned}\tag{3.14}$$

with $\partial_z=(\partial_a+i\partial_b)/2$. In particular, we have

$$\begin{aligned}-\operatorname{Im} q_0(x,\frac{2}{i}\partial_x\Psi_t(x)) &\geq \sum_{j=1}^d \lambda_j t(a_j^2+b_j^2)+\mathcal{O}(t^2(a,b)^2+t(a,b)^3)_{|_{a-ib=\kappa_U^{-1}(x)}}\\ &\geq \frac{t}{C_0}|x|^2,\end{aligned}\tag{3.15}$$

for $0<t$ and x small enough. Therefore, since $z(h)\in D(0,C_0h)$, we obtain

$$\begin{aligned}-\operatorname{Im}&\langle\chi(Q-z(h))u,u\rangle_{H_{\Psi_t}(\kappa_U(\Omega_3))}\\ &\geq\langle\frac{t}{C}|x|^2u,u\rangle_{H_{\Psi_t}(\kappa_U(\Omega_4))}+\mathcal{O}(h)\|u\|^2_{H_{\Psi_t}(\kappa_U(\Omega_2))}\\ &\geq h\|u\|^2_{H_{\Psi_t}(\kappa_U(\Omega_4))}+\mathcal{O}(h)\|u\|^2_{H_{\Psi_t}(\kappa_U(\Omega_2\setminus\Omega_4))}+\mathcal{O}(h)\|u\|^2_{H_{\Psi_t}(\{|x|<C_1\sqrt{h}\})}.\end{aligned}\tag{3.16}$$

For $n\in\mathbb{N}$, we denote by $\tau_n: H_{\Psi_t}(\kappa_U(\Omega_2))\to H_{\Psi_t}(\kappa_U(\Omega_2))$ the operator defined as

$$\tau_n(v)=\sum_{|\alpha|<n}\frac{1}{\alpha!}\partial_x^\alpha v(0)x^\alpha,\tag{3.17}$$

and we recall the following

Lemma 3.2 ([11], Lemme 4.5)**.** *Let $0<C_1<C_2$ be fixed constants. There exists a sequence $(c_n)_n$ of real positive numbers such that $c_n\to 0$ as $n\to+\infty$, and, for any $n\in\mathbb{N}$, for any $v\in H_{\Psi_t}(\kappa_U(\Omega_2))\cap\operatorname{Ker}\tau_n$ it holds that,*

$$\|v\|_{H_{\Psi_t}(\{|x|<C_1\sqrt{h}\})}\leq c_n\|v\|_{H_{\Psi_t}(\{|x|<C_2\sqrt{h}\})}.\tag{3.18}$$

Writing (3.16) for $u\in\operatorname{Ker}\tau_n$ and n large enough, we obtain

$$-\operatorname{Im}\langle\chi(Q-z)u,u\rangle_{H_{\Psi_t}(\kappa_U(\Omega_3))}\geq h\|u\|^2_{H_{\Psi_t}(\kappa_U(\Omega_4))}+\mathcal{O}(h)\|u\|^2_{H_{\Psi_t}(\kappa_U(\Omega_2\setminus\Omega_4))},\tag{3.19}$$

so that

$$h\|u\|_{H_{\Psi_t}(\kappa_U(\Omega_4))}\leq\|(Q-z)u\|_{H_{\Psi_t}(\kappa_U(\Omega_3))}+\mathcal{O}(h)\|u\|_{H_{\Psi_t}(\kappa_U(\Omega_2\setminus\Omega_4))}.\tag{3.20}$$

Now, we come back to the initial problem and we suppose that the assumptions of Theorem 2.2 hold. Since $(P - z(h))u = 0$ analytically microlocally in Ω_0, we have, for $0 < t$ small enough,

$$(Q - z)U\mathcal{T}u = \mathcal{O}(e^{-\varepsilon/h}), \tag{3.21}$$

in $H_{\Psi_t}(\kappa_U(\Omega_2))$ for some $\varepsilon > 0$. Notice that the constant ε may change from line to line in what follows, and depends on t. Applying $1 - \tau_n$, we obtain

$$\begin{aligned}(Q - z)(1 - \tau_n)U\mathcal{T}u &= [\tau_n, Q]U\mathcal{T}u + \mathcal{O}(e^{-\varepsilon/h}) \\ &= \mathcal{O}(h^{3/2})\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2))} + \mathcal{O}(e^{-\varepsilon/h}),\end{aligned} \tag{3.22}$$

in $H_{\Psi_t}(\kappa_U(\Omega_3))$. Here, we have used the fact that $\tau_n = \mathcal{O}(1)$ and $[\tau_n, Q] = \mathcal{O}(h^{3/2})$ thanks to Proposition 4.3 in [11] and Proposition 3.3 in [26]. Then, we get from (3.20) the estimate

$$\begin{aligned}\|(1 - \tau_n)U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_4))} \leq \mathcal{O}(e^{-\varepsilon/h}) &+ \mathcal{O}(h^{1/2})\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2))} \\ &+ \mathcal{O}(1)\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2\setminus\Omega_4))}.\end{aligned} \tag{3.23}$$

On the other hand, applying τ_n to (3.21), we get also

$$\tau_n(Q - z)\tau_n U\mathcal{T}u + \tau_n Q(1 - \tau_n)U\mathcal{T}u = \mathcal{O}(e^{-\varepsilon/h}). \tag{3.24}$$

Now, if we set

$$\widetilde{Q} = \sum_{j=1}^{d} \lambda_j x_j hD_{x_j} - \frac{ih}{2}\sum_{j=1}^{d} \lambda_j, \tag{3.25}$$

we see, with Proposition 3.3 of [26], that $Q - \widetilde{Q} = \mathcal{O}(h^{3/2})$ and $\tau_n Q(1 - \tau_n) = \mathcal{O}(h^{3/2})$ as operators from $H_{\Psi_t}(\kappa_U(\Omega_2))$ to $H_{\Psi_t}(\kappa_U(\Omega_4))$. Therefore (3.24) gives

$$\tau_n(\widetilde{Q} - z)\tau_n U\mathcal{T}u = \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(h^{3/2})\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2))}, \tag{3.26}$$

in $H_{\Psi_t}(\kappa_U(\Omega_4))$. On $\operatorname{Ran}\tau_n$, in the basis $(x^\alpha)_{|\alpha|<n}$, the operator $\tau_n(\widetilde{Q} - z)\tau_n$ reduces to the diagonal matrix with entries $(-hi\sum(\alpha_j + 1/2)\lambda_j - z)_{|\alpha|<n}$. So, if $d(z, \Gamma(h)) > \nu h$, $\tau_n(\widetilde{Q} - z)\tau_n$ is invertible on $\operatorname{Ran}\tau_n$, and its inverse is $\mathcal{O}(h^{-1})$. Then (3.26) gives

$$\tau_n U\mathcal{T}u = \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(h^{1/2})\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2))}, \tag{3.27}$$

in $H_{\Psi_t}(\kappa_U(\Omega_4))$. Adding (3.23) and (3.27), we obtain, for h small enough,

$$\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_4))} \leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(1)\|U\mathcal{T}u\|_{H_{\Psi_t}(\kappa_U(\Omega_2\setminus\Omega_4))}. \tag{3.28}$$

Then, we have, after shrinking Ω_4,

$$\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_2)} \leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(1)\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_2\setminus\Omega_4)}. \tag{3.29}$$

Using the same kind of estimates as in Proposition 3.1, one can see that

$$\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_1\setminus\Omega_4)} \leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(h^{1/2})\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_0)}. \tag{3.30}$$

Adding (3.29) and (3.30), we obtain

$$\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_1)} \leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(h^{1/2})\|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_0\setminus\Omega_1)}. \tag{3.31}$$

On the other hand, from the definition of G (see (3.8)), one can see that there exist $C > 0$ and $\varepsilon_1 > 0$, such that

$$e^{-tG/h} = \begin{cases} \mathcal{O}(e^{Ct/h}) & \text{on } \Omega_0, \\ \mathcal{O}(e^{-\varepsilon_1 t/h}) & \text{on } A_+. \end{cases} \tag{3.32}$$

Moreover, for each $\varepsilon_2 > 0$ there exists $\omega \subset \Omega_1$, a small enough neighborhood of 0 such that, in ω, we have

$$e^{-tG/h} \geq e^{-\varepsilon_2 t/h}. \tag{3.33}$$

Then (3.31) gives

$$\begin{aligned} e^{-\varepsilon_2 t/2h} \|\mathcal{T}u\|_{H_\Phi(\omega)} &\leq \|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_1)} \\ &\leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(h^{1/2}) \|\mathcal{T}u\|_{H_{\Phi+tG}(\Omega_0 \backslash \Omega_1)} \\ &\leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(e^{-\varepsilon_1 t/h}) \|\mathcal{T}u\|_{H_\Phi(A_+)} + \mathcal{O}(e^{tC/h}) \|\mathcal{T}u\|_{H_\Phi(A_- \cup A_0)} \\ &\leq \mathcal{O}(e^{-\varepsilon/h}) + \mathcal{O}(e^{-\varepsilon_1 t/h}) + \mathcal{O}(e^{tC/h} e^{-\delta/h}), \end{aligned} \tag{3.34}$$

since $\|\mathcal{T}u\|_{H_\Phi(\mathbb{C}^n)} = \|u\|_{L^2(\mathbb{R}^n)} \leq 1$. Choosing first $t > 0$ small enough and then ε_2 small enough, we get

$$\|\mathcal{T}u\|_{H_\Phi(\omega)} = \mathcal{O}(e^{-\widetilde{\delta}/h}), \tag{3.35}$$

for some $\widetilde{\delta} > 0$, and Theorem 2.2 follows.

4. Uniqueness in the $\mathcal{C}^\infty$ case

This section is devoted to the proof of Theorem 2.1. Let us recall briefly the assumptions (2.7)–(2.10): We suppose that $P = \mathrm{Op}_h(p)$, where p is a real valued $\mathcal{C}^\infty$ function, depending on the parameter $h \in]0, 1]$ say, in a fixed neighborhood of $(0, 0)$ in $T^*\mathbb{R}^d$. We also assume that p has an asymptotic expansion with respect to h:

$$p(x, \xi, h) \sim \sum_{k \geq 0} p_k(x, \xi) h^k, \tag{4.1}$$

and that

$$p_0(x, \xi) = \sum_{j=1}^{d} \frac{\lambda_j}{2} (\xi_j^2 - x_j^2) + \mathcal{O}((x, \xi)^3), \tag{4.2}$$

where the λ_j's are real and positive numbers. Finally, we assume that $z \in D(0, C_0 h)$ for some $C_0 > 0$.

Recalling the discussion in Section 2.2, and since Λ_+ and Λ_- are Lagrangian manifolds, one can choose local symplectic coordinates (y, η) such that

$$p_0(x, \xi) = B(y, \eta) y \cdot \eta, \tag{4.3}$$

where $(y,\eta)\mapsto B(y,\eta)$ is a $\mathcal{C}^\infty$ mapping from a neighborhood of (0,0) in $T^*\mathbb{R}^d$ to the space $\mathcal{M}_d(\mathbb{R})$ of $d\times d$ matrices with real entries such that, using the notations of Section 2,

$$B(0,0)=\begin{pmatrix}\lambda_1/2 & & \\ & \ddots & \\ & & \lambda_d/2\end{pmatrix}. \tag{4.4}$$

Now if U is a unitary Fourier Integral Operator (FIO) microlocally defined in a neighborhood of $(0,0)$, whose canonical transformation is the map $(x,\xi)\mapsto(y,\eta)$, we denote

$$\widehat{P}=UPU^{-1}. \tag{4.5}$$

Then $\widehat{P}$ is a pseudodifferential operator, with a real (modulo $\mathcal{O}(h^\infty)$) symbol $\widehat{p}(y,\eta)=\sum_j\widehat{p}_j(y,\eta)h^j$, and such that

$$\widehat{p}_0=B(y,\eta)y\cdot\eta. \tag{4.6}$$

In order to turn our microlocal problem into a global one, we extend our symbol p as a smooth function on the whole $T^*\mathbb{R}^d$. Notice that this idea cannot be used in the analytic category. The way we perform this extension is reminiscent of the so-called Complex Absorption Potential Method, used by quantum chemists, and mathematically studied in a paper by P. Stefanov [30].

In the following, the notation $f\prec g$ means that $g=1$ near the support of f. Let χ_5, $\chi_8\in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d))$ be such that the support of χ_8 is a small enough neighborhood of 0 and $\mathbf{1}_{\{0\}}\prec\chi_5\prec\chi_8$. We define

$$\widetilde{p}(y,\eta)=\widehat{p}(y,\eta)\chi_8(y,\eta)-i\sqrt{h}(1-\chi_5(y,\eta)), \tag{4.7}$$

and we also denote $\tilde{P}=\mathrm{Op}_h(\tilde{p})$. Let us mention that, as one can see following the proof, one could have taken h^ε with $0<\varepsilon<1$ instead of $\sqrt{h}$ in front of the $1-\chi_5$ term.

Now we choose $\chi_7\in C_0^\infty(\mathrm{T}^*\mathbb{R}^d)$ with $\chi_5\prec\chi_7\prec\chi_8$, and we set

$$g_1(y,\eta)=(y^2-\eta^2)\chi_7(y,\eta)\ln(1/h). \tag{4.8}$$

Notice that $H_{\hat{p}_0}g_1(y,\eta)>0$ for any $(y,\eta)\neq(0,0)$. Following the appendix of [3], we also define

$$g_2(y,\eta)=\Big(\ln\Big\langle\frac{y}{\sqrt{hM}}\Big\rangle-\ln\Big\langle\frac{\eta}{\sqrt{hM}}\Big\rangle\Big)\chi_3(y,\eta). \tag{4.9}$$

where $M>0$ will be fixed later and $\chi_3\in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d))$ is such that $\mathbf{1}_{\{0\}}\prec\chi_3\prec\chi_5$. For t_1, $t_2>0$ we set

$$G_{\pm1}=\mathrm{Op}_h(e^{\pm t_1g_1(y,\eta)})\text{ and }G_{\pm2}=\mathrm{Op}_h(e^{\pm t_2g_2(y,\eta)}), \tag{4.10}$$

and we see that these h-pseudodifferential operators satisfy $G_{\pm1}\in\Psi_h^0(h^{-Ct_1})$ as well as $G_{\pm2}\in\Psi_h^{1/2}(h^{-Ct_2})$, for some $C>0$.

Now, as in N. Dencker, J. Sjöstrand and M. Zworski [8, Section 4], or in the very recent paper [4], we set

$$\begin{aligned}Q_z&=G_{-2}G_{-1}(\widetilde{P}-z)G_1G_2\\&=G_{-2}G_{-1}\Big(\mathrm{Op}_h(\widehat{p}\chi_8)+i\sqrt{h}(1-\chi_5)+z\Big)G_1G_2,\end{aligned} \tag{4.11}$$

and we consider each term of the above sum separately.

• First of all, we consider the operator $G_{-1}\operatorname{Op}_h\big(i\sqrt{h}(1-\chi_5)\big)G_1$. By symbolic calculus in the class $\Psi_h^0(1)$ (see Proposition A.1), writing $F=\operatorname{Op}_h\big((1-\chi_5)\big)$, we have $FG_1=\operatorname{Op}_h(\varphi_1)$ with, for any $N_1\in\mathbb{N}$,

$$\begin{aligned}\varphi_1(x,\xi)=&\sum_{k=0}^{N_1}\frac{1}{k!}\Big(\big(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\big)^k(1-\chi_5)(x,\xi)e^{t_1g_1(y,\eta)}\Big)\Big|_{y=x,\eta=\xi}\\ &+h^{N_1-Ct_1}\mathcal{S}_h^0(1).\end{aligned}\tag{4.12}$$

Then again, $G_{-1}\operatorname{Op}_h(\varphi_1)=\operatorname{Op}_h(\varphi_0)$ with

$$\begin{aligned}\varphi_0(x,\xi)=&\sum_{k=0}^{N_0}\frac{1}{k!}\Big(\big(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\big)^k e^{-t_1g_1(x,\xi)}\varphi_1(y,\eta)\Big)\Big|_{y=x,\eta=\xi}\\ &+h^{N_0-2Ct_1}\mathcal{S}_h^0(1).\end{aligned}\tag{4.13}$$

But it is easy to see that the k-th term in the sum (4.13) is $\mathcal{O}(h^k)$, and choosing $N_0\in\mathbb{N}$ such that $N_0-2Ct_1\geq 0$, we get that $G_1FG_{-1}\in\Psi_h^0(1)$. We also see on (4.13) that the symbol of G_1FG_{-1} is supported inside the support of $1-\chi_5$ modulo $\mathcal{O}(h^\infty)$.

Now since $\chi_3\prec\chi_5$, we also have, using the same kind of arguments, but in the class $\Psi_h^{1/2}(1)$, that

$$G_{-2}G_{-1}\operatorname{Op}_h\big(i\sqrt{h}(1-\chi_5(y,\eta))\big)G_1G_2=G_{-1}\operatorname{Op}_h\big(i\sqrt{h}(1-\chi_5(y,\eta))\big)G_1+\mathcal{O}(h^\infty).$$

Notice that without explicit notification, any error term in equalities between pseudodifferential operator has to be understood in the sense of bounded operators on L^2. Finally, keeping only the first term in the expansion (4.13), we get

$$G_{-2}G_{-1}\operatorname{Op}_h\big(i\sqrt{h}(1-\chi_5(y,\eta))\big)G_1G_2=\operatorname{Op}_h\big(i\sqrt{h}(1-\chi_5(y,\eta))\big)+O\big(h^{3/2}\big).\tag{4.14}$$

• We consider now the second term in (4.11), and we set

$$\widehat{Q}=G_{-1}\operatorname{Op}_h\big(\widehat{p}(y,\eta)\chi_8(y,\eta)\big)G_1.\tag{4.15}$$

We obtain again by symbolic calculus in the class $\Psi_h^0(1)$ that

$$\widehat{Q}=\operatorname{Op}_h(\widehat{q})+\mathcal{O}(h^\infty),\tag{4.16}$$

where $\widehat{q}(y,\eta)\in\mathcal{S}_h^0(1)$ is supported inside the support of χ_8 and satisfies

$$\widehat{q}=\widehat{p}_0\chi_8+h\widehat{p}_1\chi_8+iht_1\{g_1,\widehat{p}_0\chi_8\}+h^2\ln^2(1/h)\mathcal{S}_h^0(1).\tag{4.17}$$

As in [3], since $G_{\pm 2}$ is in some $\Psi_h^{1/2}$, we need to rescale the variables in order to compute the symbol of $G_{-2}\widehat{Q}G_2$: We define a unitary transformation V on $L^2(\mathbb{R}^d)$ by

$$Vf(y)=\lambda^{-d/2}f\big(\lambda^{-1}y\big),\quad \lambda=\sqrt{hM},\tag{4.18}$$

and, if $a(y,\eta,h)$ is a family of distributions in $\mathcal{S}'(\mathrm{T}^*(\mathbb{R}^d))$, we have

$$V^{-1}\operatorname{Op}_h\big(a(y,\eta,h)\big)V=\operatorname{Op}_{\frac{1}{M}}\big(a\big(\lambda Y,\lambda H,\frac{\lambda^2}{M}\big)\big).\tag{4.19}$$

Notice that here and in what follows, we always assume that $\lambda\ll 1$.

Then we set $\widetilde{Q}=V^{-1}G_{-2}\widehat{Q}G_2V$ and we notice that

$$\widetilde{Q}=\operatorname{Op}_{\frac{1}{M}}\big(e^{-t_2\widetilde{g}_2(Y,H)}\big)\operatorname{Op}_{\frac{1}{M}}\big(\widehat{q}(\lambda Y,\lambda H)\big)\operatorname{Op}_{\frac{1}{M}}\big(e^{t_2\widetilde{g}_2(Y,H)}\big)+\mathcal{O}(h^\infty),\tag{4.20}$$

where

$$\widetilde{g}_2(Y,H) = (\ln\langle Y\rangle - \ln\langle H\rangle)\,\chi_3\big(\lambda(Y,H)\big). \tag{4.21}$$

We notice that, for any $\alpha,\beta\in\mathbb{N}^d$, and for some constants $C_{\alpha,\beta}$ and C that are independent of λ,

$$\left|\partial_Y^\alpha\partial_H^\beta e^{\pm t_2\widetilde{g}_2(Y,H)}\right| \le C_{\alpha,\beta}\left\langle\frac{\langle Y\rangle}{\ln\langle Y\rangle}\right\rangle^{-|\alpha|}\left\langle\frac{\langle H\rangle}{\ln\langle H\rangle}\right\rangle^{-|\beta|}\langle (Y,H)\rangle^{Ct_2}, \tag{4.22}$$

Using (4.6), and since $\lambda\langle(Y,H)\rangle$ can be considered as $\mathcal{O}(1)$ for $\hat{p}_0\chi_8$ is compactly supported, we see also that, for any $\alpha,\beta\in\mathbb{N}^d$,

$$\left|\partial_Y^\alpha\partial_H^\beta(\widehat{p}_0\chi_8)(\lambda Y,\lambda H)\right| \le C_{\alpha,\beta}\lambda^2\langle (Y,H)\rangle^{2-|\alpha|-|\beta|}, \tag{4.23}$$

At this point, it is convenient to introduce a new class of symbols: We shall write that $f(Y,H,\frac{1}{M})$ belongs to $\widetilde{\mathcal{S}}_{\frac{1}{M}}(m)$ if it is a smooth function of (Y,H) such that, for any $\alpha,\beta\in\mathbb{N}^d$, there exists a constant $C_{\alpha,\beta}>0$ such that

$$|\partial_Y^\alpha\partial_H^\beta f(Y,H,\frac{1}{M})| \le C_{\alpha,\beta}\langle Y\rangle^{-|\alpha|/2}\langle H\rangle^{-|\beta|/2}m(Y,H). \tag{4.24}$$

Here the function m is any order function in the sense of [9], Chapter 7 (see also Appendix A). With these notations, we have $e^{-t_2\widetilde{g}_2}\in\widetilde{\mathcal{S}}_{\frac{1}{M}}(\langle(Y,H)\rangle^{Ct_2})$ and $(Y,H)\mapsto\widehat{p}_0\chi_8(\lambda Y,\lambda H)\in\widetilde{\mathcal{S}}_{\frac{1}{M}}(\lambda^2\langle(Y,H)\rangle^2)$, uniformly with respect to λ. Notice also that if $a(y,\eta,h)\in\mathcal{S}_h^0(1)$ for example, then $a\big(\lambda Y,\lambda H,\frac{\lambda^2}{M}\big)\in\widetilde{\mathcal{S}}_{\frac{1}{M}}(1)$.

Now we compute the symbol of $\widetilde{Q}$, and we shall again consider each term in (4.17) separately.

From M^{-1}-pseudodifferential calculus for symbols in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we get that

$$\operatorname{Op}_{\frac{1}{M}}\left(e^{-t_2\widetilde{g}_2}\right)\operatorname{Op}_{\frac{1}{M}}(\widehat{p}_0\chi_8(\lambda Y,\lambda H)) = \operatorname{Op}_{\frac{1}{M}}(\widetilde{\ell}_0), \tag{4.25}$$

where

$$\begin{aligned}\widetilde{\ell}_0 =& e^{-t_2\widetilde{g}_2}\Big(\widehat{p}_0\chi_8 + i\frac{t_2}{2M}\{\widetilde{g}_2,\widehat{p}_0\chi_8\}\Big) - \frac{1}{8M^2}\Big(\partial_Y^2 e^{-t_2\widetilde{g}_2}\partial_H^2(\widehat{p}_0\chi_8)\\ &- 2\partial_{Y,H}^2 e^{-t_2\widetilde{g}_2}\partial_{Y,H}^2(\widehat{p}_0\chi_8) + \partial_H^2 e^{-t_2\widetilde{g}_2}\partial_Y^2(\widehat{p}_0\chi_8)\Big)\\ &+ e^{-t_2\widetilde{g}_2}\widetilde{\mathcal{S}}_{\frac{1}{M}}(M^{-3}\lambda^2) + \widetilde{\mathcal{S}}_{\frac{1}{M}}\big(\lambda^2 M^{-\infty}\langle(Y,H)\rangle^{-\infty}\big).\end{aligned} \tag{4.26}$$

Notice that we have used (4.23) for the first error term above. Using the particular form of p_0 in (4.3) and that of $\widetilde{g}_2$ in (4.21), and the fact that $\lambda\langle(Y,H)\rangle=\mathcal{O}(1)$ since $\hat{p}_0\chi_8$ is compactly supported, we obtain, for some $\varepsilon>0$,

$$\begin{aligned}\widetilde{\ell}_0 =& e^{-t_2\widetilde{g}_2}\Big(\widehat{p}_0\chi_8 + i\frac{t_2}{2M}\{\widetilde{g}_2,\widehat{p}_0\chi_8\}\Big)\\ &+ e^{-t_2\widetilde{g}_2}\big(\widetilde{\mathcal{S}}_{\frac{1}{M}}(t_2^2\lambda^2M^{-2}) + \mathcal{O}_M(h^{1+\varepsilon})\widetilde{\mathcal{S}}_{\frac{1}{M}}(1) + \widetilde{\mathcal{S}}_{\frac{1}{M}}(M^{-3}\lambda^2)\big)\\ &+ \widetilde{\mathcal{S}}_{\frac{1}{M}}\big(\lambda^2M^{-\infty}\langle(Y,H)\rangle^{-\infty}\big).\end{aligned} \tag{4.27}$$

Here, the notation $\mathcal{O}_M(m)$ means that the function is bounded by m with bound depending on M. Now we compute the symbol $\tilde{q}_0$ defined by

$$\operatorname{Op}_{\frac{1}{M}}(\widetilde{\ell}_0)\operatorname{Op}_{\frac{1}{M}}\big(e^{t_2\widetilde{g}_2}\big)=\operatorname{Op}_{\frac{1}{M}}(\tilde{q}_0). \tag{4.28}$$

We have

$$\begin{aligned}\widetilde{q}_0 =&\widehat{p}_0\chi_8+i\frac{t_2}{2M}\{\widetilde{g}_2,\widehat{p}_0\chi_8\}-\frac{it_2}{2M}\Big\{\widehat{p}_0\chi_8+i\frac{t_2}{2M}\{\widetilde{g}_2,\widehat{p}_0\chi_8\},\widetilde{g}_2\Big\}\\ &+\widetilde{\mathcal{S}}_{\frac{1}{M}}(t_2^2\lambda^2M^{-2})+\mathcal{O}_M(h^{1+\varepsilon})\widetilde{\mathcal{S}}_{\frac{1}{M}}(1)+\widetilde{\mathcal{S}}_{\frac{1}{M}}(M^{-3}\lambda^2)+\widetilde{\mathcal{S}}_{\frac{1}{M}}\big(\lambda^2M^{-\infty}\big)\\ =&\Big(\widehat{p}_0\chi_8+i\frac{t_2}{M}\{\widetilde{g}_2,\widehat{p}_0\chi_8\}\Big)+\widetilde{\mathcal{S}}_{\frac{1}{M}}(t_2^2\lambda^2M^{-2})+O_M(h^{1+\varepsilon})\widetilde{\mathcal{S}}_{\frac{1}{M}}(1)+\widetilde{\mathcal{S}}_{\frac{1}{M}}(M^{-3}\lambda^2).\end{aligned} \tag{4.29}$$

Notice that we have used the following explicit expression:

$$\begin{aligned}\{\widetilde{g}_2,\widehat{p}_0\chi_8\}=&-\lambda^2\chi_3\big(\lambda(Y,H)\big)\Big(\frac{H}{\langle H\rangle^2}\cdot\big(B(\lambda Y,\lambda H)H+\partial_yB(\lambda Y,\lambda H)\lambda Y\cdot H\big)\\ &+\frac{Y}{\langle Y\rangle^2}\cdot\big(B(\lambda Y,\lambda H)Y+\partial_\eta B(\lambda Y,\lambda H)\lambda Y.H\big)\Big)\\ &+\lambda^2\left(\ln\langle Y\rangle-\ln\langle H\rangle\right)\{\chi_3,\widehat{p}_0\chi_8\}\big(\lambda(Y,H)\big),\end{aligned} \tag{4.30}$$

so that, in particular, we have written $\{\widetilde{g}_2,\{\widetilde{g}_2,\widehat{p}_0\chi_8\}\}=\mathcal{O}(\lambda^2)$ in (4.29).

Now we compute the contribution of the second term in (4.17). Let us define the symbol $\tilde{\ell}_1$ by

$$\operatorname{Op}_{\frac{1}{M}}\big(e^{-t_2\widetilde{g}_2}\big)\operatorname{Op}_{\frac{1}{M}}\big((h\widehat{p}_1\chi_8)(\lambda Y,\lambda H)\big)=\operatorname{Op}_{\frac{1}{M}}(\tilde{\ell}_1). \tag{4.31}$$

We have first

$$\operatorname{Op}_h(e^{-t_2g_2})\operatorname{Op}_h(h\widehat{p}_1\chi_8)=\operatorname{Op}_h(\ell_1), \tag{4.32}$$

where

$$\ell_1=he^{-t_2g_2}\widehat{p}_1\chi_8+\frac{h^2}{\lambda}e^{-t_2g_2}\mathcal{S}_h^{1/2}(1)+h^\infty\mathcal{S}_h^{1/2}(1). \tag{4.33}$$

Restoring the (Y,H) variables, we obtain, also since $h^2/\lambda\leq\lambda h$,

$$\tilde{\ell}_1(Y,H)=he^{-t_2\widetilde{g}_2}\widehat{p}_1\chi_8(\lambda Y,\lambda H)+e^{-t_2\widetilde{g}_2}\widetilde{\mathcal{S}}_{\frac{1}{M}}(\lambda h)+\widetilde{\mathcal{S}}_{\frac{1}{M}}(\lambda^\infty). \tag{4.34}$$

Then, using the symbolic calculus in the class $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we get

$$\begin{aligned}\operatorname{Op}_{\frac{1}{M}}(\tilde{q}_1):=&\operatorname{Op}_{\frac{1}{M}}(\tilde{\ell}_1)\operatorname{Op}_{\frac{1}{M}}\big(e^{t_2\widetilde{g}_2}\big)\\ =&\operatorname{Op}_{\frac{1}{M}}(h\widehat{p}_1\chi_8)+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{1+\varepsilon}).\end{aligned} \tag{4.35}$$

As for the third term in (4.17), we write

$$\operatorname{Op}_{\frac{1}{M}}\big(e^{-t_2\widetilde{g}_2}\big)\operatorname{Op}_{\frac{1}{M}}\big((iht_1\{g_1,\widehat{p}_0\chi_8\})(\lambda Y,\lambda H)\big)=\operatorname{Op}_{\frac{1}{M}}(\tilde{\ell}_2), \tag{4.36}$$

with

$$\tilde{\ell}_2=iht_1e^{-t_2\widetilde{g}_2}\{g_1,\widehat{p}_0\chi_8\}+e^{-t_2\widetilde{g}_2}\lambda h\ln(1/h)\widetilde{\mathcal{S}}_{\frac{1}{M}}(1)+\mathcal{O}(h^\infty). \tag{4.37}$$

Then we remark that, for any $\alpha,\beta\in\mathbb{N}^d$, we have, for some $C_{\alpha,\beta}>0$,

$$\big|\partial_Y^\alpha\partial_H^\beta\big(iht_1\{g_1,\widehat{p}_0\chi_8\}\big)\big|\leq C_{\alpha,\beta}\lambda h\ln(1/h)\langle(Y,H)\rangle^{1-|\alpha|-|\beta|}, \tag{4.38}$$

so that the function $(Y,H)\mapsto iht_1\{g_1,\widehat{p}_0\chi_8\}(\lambda Y,\lambda H)$ belongs to $\widetilde{\mathcal{S}}_{\frac{1}{M}}(\lambda h\ln(1/h)\langle(Y,H)\rangle)$. Therefore, using (4.23) and the symbolic calculus in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we have

$$\begin{aligned}\operatorname{Op}_{\frac{1}{M}}(\tilde q_2) &:= \operatorname{Op}_{\frac{1}{M}}(\tilde\ell_2)\operatorname{Op}_{\frac{1}{M}}\left(e^{t_2\widetilde g_2}\right)\\ &= \operatorname{Op}_{\frac{1}{M}}\left(iht_1\{g_1,\widehat{p}_0\chi_8\}\right)+\mathcal{O}(\lambda h\ln(1/h)).\end{aligned}\tag{4.39}$$

Finally, let $r(y,\eta)$ be the remainder term in (4.17). We see that in $r\in\mathcal{S}_h^0(h^{3/2})$, and that r has compact support inside the support of χ_8. In the variables (Y,H), we have

$$\left|\partial_Y^\alpha\partial_H^\beta r(\lambda Y,\lambda H)\right|\le C_{\alpha,\beta}\,h^2\ln^2(1/h)\langle(Y,H)\rangle^{-|\alpha|-|\beta|}.\tag{4.40}$$

Therefore, working again in the class $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we obtain

$$\operatorname{Op}_{\frac{1}{M}}\left(e^{-t_2\widetilde g_2}\right)\operatorname{Op}_{\frac{1}{M}}\left(r(\lambda Y,\lambda H)\right)\operatorname{Op}_{\frac{1}{M}}\left(e^{t_2\widetilde g_2}\right)=\mathcal{O}(h^2\ln^2(1/h)).\tag{4.41}$$

- It remains to study $G_{-2}G_{-1}zG_1G_2$. First of all, since $\{e^{-t_1\widetilde g_1},e^{t_1\widetilde g_1}\}=0$, we have

$$G_{-1}zG_1=\operatorname{Op}_h(z(1+\mathcal{S}_h^0(h^2\ln^2(1/h))).\tag{4.42}$$

Then working in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we obtain

$$G_{-2}G_{-1}zG_1G_2=z+\mathcal{O}(zM^{-2}).\tag{4.43}$$

Finally, collecting (4.11), (4.14), (4.17), (4.29), (4.35), (4.39), (4.41), we have obtained that

$$\begin{aligned}Q_z=&\operatorname{Op}_h\left(\widehat{p}_0\chi_8+h\widehat{p}_1\chi_8+iht_1\{g_1,\widehat{p}_0\chi_8\}\right)\\ &+\operatorname{Op}_h\Big(it_2M^{-1}\{\widetilde g_2,\widehat{p}_0\chi_8\}-i\sqrt{h}(1-\chi_5)\Big)-z\\ &+\mathcal{O}(t_2^2hM^{-1})+\mathcal{O}_M(h^{1+\varepsilon})+\mathcal{O}(hM^{-2}),\end{aligned}\tag{4.44}$$

and we are able to prove the following

Proposition 4.1. *Let δ, $C_0>0$, $t_1\gg1$ and $t_2\gg1$ be fixed. For M^{-1} fixed and h both small enough, we have:*

i) For $z\in D(0,C_0h)$ and $\operatorname{Im}z>\delta h$, the operator $Q_z:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is invertible and

$$\|Q_z^{-1}\|=\mathcal{O}(h^{-1}).\tag{4.45}$$

ii) There exists an operator $K=K(h)$ with $\operatorname{Rank}K=\mathcal{O}(1)$ and $K=\mathcal{O}(1)$ such that $Q_z+hK:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is invertible for $z\in D(0,C_0h)$ and

$$\|(Q_z+hK)^{-1}\|=\mathcal{O}(h^{-1}).\tag{4.46}$$

Proof. For $u\in\mathcal{S}(\mathbb{R}^d)$, we have using (4.44),

$$\begin{aligned}-\operatorname{Im}\langle Q_zu,u\rangle_{L^2(\mathbb{R}^d)}\ge&-\Big\langle\Big(\operatorname{Op}_h\Big(ht_1\{g_1,\widehat{p}_0\chi_8\}+\frac{t_2}{M}\{\widetilde g_2,\widehat{p}_0\chi_8\}-\sqrt{h}(1-\chi_5)\Big)-\operatorname{Im}z\Big)u,u\Big\rangle\\ &+\big(\mathcal{O}(t_2^2hM^{-1})+\mathcal{O}_M(h^{1+\varepsilon})+\mathcal{O}(hM^{-2})\big)\|u\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{4.47}$$

Let χ_2, $\varphi_1 \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d);[0,1])$ be such that $\mathbf{1}_{\{0\}} \prec \chi_2 \prec \chi_3$, $\varphi_1 = 0$ near $(0,0)$, and $(1-\chi_2)\chi_3 \prec \varphi_1 \prec \chi_5$. From (4.30), and since χ_2 vanishes on the support of $\{\chi_3, \widehat{p}_0\chi_8\}$, we have, for some $\varepsilon > 0$,

$$-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2(\lambda Y, \lambda H) \begin{cases} \in \mathcal{S}^0_{\frac{1}{M}}(h) \\ \geq \varepsilon h \Big(\dfrac{Y^2}{\langle Y\rangle^2} + \dfrac{H^2}{\langle H\rangle^2}\Big)\chi_2^2(\lambda Y, \lambda H), \end{cases} \tag{4.48}$$

if the support of χ_3 is small enough. On the other hand, using again the fact that $\lambda(Y,H) = \mathcal{O}(1)$, we notice that

$$-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}(1-\chi_2^2)(\lambda Y, \lambda H) \in \mathcal{S}^0_{\frac{1}{M}}\big(h\ln(1/h)\big). \tag{4.49}$$

Working in the variables (Y,H), using (4.48) and Gårding's inequality, we get

$$\Big\langle \operatorname{Op}_h\Big(-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2\Big)u, u\Big\rangle \geq -\frac{Ch}{M}\|u\|^2. \tag{4.50}$$

Now, since $1-\varphi_1$ and $(1-\chi_2^2)\chi_3$ have disjoint supports, we have

$$\begin{aligned} &\operatorname{Op}_h\Big(-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}(1-\chi_2^2)\Big) \\ &\qquad = \operatorname{Op}_h(\varphi_1)\operatorname{Op}_h\Big(-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}(1-\chi_2^2)\Big)\operatorname{Op}_h(\varphi_1) + \mathcal{O}(h^\infty), \end{aligned} \tag{4.51}$$

and we get from (4.49) and Calderòn–Vaillancourt's theorem, that

$$\Big\langle \operatorname{Op}_h\Big(-\frac{1}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}(1-\chi_2^2)\Big)u, u\Big\rangle \geq -Ch\ln(1/h)\|\operatorname{Op}_h(\varphi_1)u\|^2 + \mathcal{O}_M(h^\infty)\|u\|^2. \tag{4.52}$$

Here $C>0$ is uniform with respect to M and h.

Let χ_1, $\chi_6 \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d))$ with $\mathbf{1}_{\{0\}} \prec \chi_1 \prec \chi_2 \prec \chi_5 \prec \chi_6 \prec \chi_7$ and $\varphi_1 \prec \chi_6 - \chi_1$. Then

$$\begin{aligned} -ht_1\{g_1, \widehat{p}_0\chi_8\} &= -t_1 h\ln(1/h)\big\{(y^2-\eta^2)\chi_7(y,\eta), \chi_8(y,\eta)B(y,\eta)y\cdot\eta\big\} \\ &\begin{cases} \in \mathcal{S}^0_h(t_1 h\ln(1/h)) \\ \geq \varepsilon t_1 h\ln(1/h)(y^2+\eta^2) \text{ near the support of } \chi_6. \end{cases} \end{aligned} \tag{4.53}$$

Let $\varphi_2 \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d);[0,1])$ with $(1-\chi_6)\chi_7 \prec \varphi_2 \prec (1-\chi_5)\chi_8$. Using Gårding's inequality for symbols in $\mathcal{S}^0_h(t_1 h\ln(1/h))$, we obtain

$$\begin{aligned} \big\langle \operatorname{Op}_h\big(-ht_1\{g_1, \widehat{p}_0\chi_8\}\chi_6^2\big)u, u\big\rangle \geq{}& \varepsilon t_1 h\ln(1/h)\|\operatorname{Op}_h(\chi_6-\chi_1)u\|^2 \\ &+ \mathcal{O}(t_1 h^2\ln(1/h))\|u\|^2. \end{aligned} \tag{4.54}$$

We also have, as in (4.51–4.52),

$$\big\langle \operatorname{Op}_h\big(-ht_1\{g_1, \widehat{p}_0\chi_8\}(1-\chi_6^2)\big)u, u\big\rangle \geq -Ct_1 h\ln(1/h)\|\operatorname{Op}_h(\varphi_2)u\|^2 + \mathcal{O}(h^\infty)\|u\|^2. \tag{4.55}$$

Then, collecting (4.50), (4.52), (4.54) and (4.55), the inequality (4.47) becomes

$$\begin{aligned} -\operatorname{Im}\langle Q_z u, u\rangle \geq{}& \varepsilon t_1 h\ln(1/h)\|\operatorname{Op}_h(\chi_6-\chi_1)u\|^2 + \sqrt{h}\langle \operatorname{Op}_h(1-\chi_5)u, u\rangle + \operatorname{Im} z\|u\|^2 \\ &- Ch\ln(1/h)\|\operatorname{Op}_h(\varphi_1)u\|^2 - Ct_1 h\ln(1/h)\|\operatorname{Op}_h(\varphi_2)u\|^2 \\ &+ \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2, \end{aligned} \tag{4.56}$$

where C and ε are uniform with respect to h and M. Now, since $\chi_6 - \chi_1 = 1$ on $\operatorname{supp}\varphi_1$, and $1-\chi_5 = 1$ on $\operatorname{supp}\varphi_2$, Gårding's inequality in $\mathcal{S}^0_h(\sqrt{h})$ gives us, for any chosen t_1 large enough

$$-\operatorname{Im}\langle Q_z u, u\rangle_{L^2(\mathbb{R}^d)} \geq \operatorname{Im} z\|u\|^2 + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2. \tag{4.57}$$

Then we have

$$-\operatorname{Im}\langle Q_z u, u\rangle_{L^2(\mathbb{R}^d)} \geq \frac{\delta h}{2}\|u\|^2. \tag{4.58}$$

provided $\operatorname{Im} z \geq \delta h$ and M is fixed large enough (and h is small enough). Since $|\operatorname{Im}\langle Q_z u, u\rangle| \leq \|Q_z u\|\|u\|$, we get

$$\|Q_z u\| \geq \delta h/2\|u\|. \tag{4.59}$$

We can obtain the same way the same estimate for Q_z^*, and this finishes the proof of the first point of the proposition.

Now we consider the second point. Let $\varphi \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d);[0,1])$ be such that $\varphi = 1$ near 0. We denote

$$\widetilde{K} = C_1 \operatorname{Op}_h\Big(\varphi\Big(\frac{y}{\sqrt{Mh}}, \frac{\eta}{\sqrt{Mh}}\Big)\Big), \tag{4.60}$$

where $C_1 > 0$ is a large constant. Since its symbol is real, $\widetilde{K}$ is self-adjoint and $\widetilde{K} = \mathcal{O}(C_1)$. Recalling (4.18–4.19), we have

$$V^{-1}\widetilde{K}V = C_1 \operatorname{Op}_{\frac{1}{M}}\left(\varphi(Y,H)\right), \tag{4.61}$$

and, therefore, $\|\widetilde{K}\|_{tr} = \mathcal{O}(C_1 M^d)$ (see [9, Theorem 9.4]). Now, using (4.52), (4.54) and (4.55), we get

$$\begin{aligned}
-\operatorname{Im}\big\langle (Q_z - ih\widetilde{K})u, u\big\rangle \geq & \big\langle \operatorname{Op}_{\frac{1}{M}}\left(-t_2 M^{-1}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2 + C_1 h\varphi\right)u, u\big\rangle + \operatorname{Im} z\|u\|^2 \\
& + \varepsilon t_1 h\ln(1/h)\|\operatorname{Op}_h(\chi_6-\chi_1)u\|^2 + \sqrt{h}\,\langle \operatorname{Op}_h(1-\chi_5)u, u\rangle \\
& - Ch\ln(1/h)\|\operatorname{Op}_h(\varphi_1)u\|^2 - Ct_1 h\ln(1/h)\|\operatorname{Op}_h(\varphi_2)u\|^2 \\
& + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2.
\end{aligned} \tag{4.62}$$

Here, instead of using (4.50), we notice that, recalling (4.48), the term $-\frac{t_2}{M}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2 + C_1 h\varphi$ belongs to $\mathcal{S}^0_{\frac{1}{M}}(h)$ and satisfies

$$-t_2 M^{-1}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2 + C_1 h\varphi \geq \varepsilon\min(t_2, C_1)h\chi_2^2. \tag{4.63}$$

Thus, from Gårding's inequality in $\mathcal{S}^0_{\frac{1}{M}}(h)$, we obtain

$$\begin{aligned}
&\big\langle \operatorname{Op}_{\frac{1}{M}}\big(-t_2 M^{-1}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2 + C_1 h\varphi\big)u, u\big\rangle \\
&\qquad\qquad \geq \varepsilon\min(t_2, C_1)h\|\operatorname{Op}_h(\chi_2)u\|^2 + \mathcal{O}(hM^{-1})\|u\|^2.
\end{aligned} \tag{4.64}$$

Now, as in (4.57), the inequality (4.62) becomes

$$\begin{aligned}
-\operatorname{Im}\big\langle (Q_z - ih\widetilde{K})u, u\big\rangle \geq & \varepsilon\min(t_2, C_1)h\|\operatorname{Op}_h(\chi_2)u\|^2 + \operatorname{Im} z\|u\|^2 \\
& + \varepsilon t_1 h\ln(1/h)/2\|\operatorname{Op}_h(\chi_6-\chi_1)u\|^2 + \sqrt{h}/2\,\langle \operatorname{Op}_h(1-\chi_5)u, u\rangle \\
& + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2,
\end{aligned} \tag{4.65}$$

for t_1 large enough. Now, if t_1, t_2 and C_1 are large enough, we get as in (4.56–4.57),

$$\begin{aligned} - \operatorname{Im} \big\langle (Q_z - ih\widetilde{K})u, u \big\rangle &\geq 2C_0 h\|u\|^2 + \operatorname{Im} z\|u\|^2 + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2 \\ &\geq C_0 h\|u\|^2 + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}_M(h^{1+\varepsilon})\|u\|^2, \end{aligned} \tag{4.66}$$

for $z \in D(0, C_0 h)$. And this implies

$$\|(Q_z - ih\widetilde{K})^{-1}\| = \mathcal{O}(h^{-1}). \tag{4.67}$$

for M large enough. Since $\widetilde{K} = \mathcal{O}(1)$ is self-adjoint and $\|\widetilde{K}\|_{tr} = \mathcal{O}(1)$, one can find a bounded operator K such that $\operatorname{Rank} K$ is finite and independent of h, and $-i\widetilde{K} - K$ is as small as needed (uniformly with respect to h, M, ...), and the proposition is proved. ☐

Now we can estimate $(Q - z)^{-1}$ for z away from some discrete set $\Gamma(h)$. We follow J. Sjöstrand [27] and S. H. Tang and M. Zworski [31].

Proposition 4.2. *Suppose that the assumptions of Proposition 4.1 hold. Then there is a discrete set $\Gamma(h)$ independent of t_1, t_2 and M, with $\#\Gamma(h) \cap D(0, C_0 h) = \mathcal{O}(1)$, such that $Q_z : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is invertible for $z \notin \Gamma(h) \cap D(0, C_0 h)$.*

Moreover, if $d(z, \Gamma(h)) > \nu h^N$, for some $\nu > 0$ and $N \geq 1$, we have

$$\|Q_z^{-1}\| = \mathcal{O}(h^{-C}), \tag{4.68}$$

where C depends only on N and C_0.

Proof. We begin the proof by showing that Q_z is invertible outside a finite set $\Gamma(h)$. Here again, we use ideas developed for the study of resonances. Let

$$F(z) = \det\big(Q_z(Q_z + hK)^{-1}\big) = \det\big(1 - hK(Q_z + hK)^{-1}\big). \tag{4.69}$$

Since K is trace class, $F(z)$ is well-defined and holomorphic in $D(0, 2C_0 h)$. From (4.46), we get

$$F(z) = \mathcal{O}(1), \tag{4.70}$$

for $z \in D(0, 2C_0 h)$. On the other hand, for $\operatorname{Im} z > \delta h$, we see from (4.45) that $(F(z))^{-1} = \det\big((Q_z + hK)Q_z^{-1}\big) = \det\big(1 + hKQ_z^{-1}\big)$, so that

$$|F(z)| > \varepsilon, \tag{4.71}$$

still for $\operatorname{Im} z > \delta h$. The estimates (4.70), (4.71) and Jensen's formula imply that the number of zeros of $F(z)$ in $D(0, C_0 h)$ is bounded. Using the properties of the determinant of an analytic family of operators, we get that Q_z is invertible outside a bounded set $\Gamma(h)$ and that the algebraic multiplicity of the poles of Q_z^{-1} is bounded. At this point, $\Gamma(h)$ depends on t_1, t_2 ..., but if h and M^{-1} are small enough, we have

$$G_1 G_{-1} = 1 + \mathcal{O}(h^2 \ln^2 \frac{1}{h}), \quad G_{-1} G_1 = 1 + \mathcal{O}(h^2 \ln^2 \frac{1}{h}), \tag{4.72}$$

$$G_2 G_{-2} = 1 + \mathcal{O}(M^{-2}), \qquad G_{-2} G_2 = 1 + \mathcal{O}(M^{-2}), \tag{4.73}$$

so that G_1, G_{-1}, G_2 and G_{-2} are invertible. Thus, $\Gamma(h)$ is nothing but the set of eigenvalues of $\widetilde{P}$, which are independent of t_1, t_2 and M. These eigenvalues have finite multiplicity.

In order to estimate $\|Q_z^{-1}\|$ for z away from $\Gamma(h)$, we use the same strategy as in [27]. Let $e_1, \dots, e_N$ be an orthonormal basis of $\operatorname{Im} K^* = (\operatorname{Ker} K)^{\perp}$ and $(e_j)_{j \geq N+1}$ an orthonormal

basis of $\operatorname{Ker} K$. We denote $R_+ : L^2(\mathbb{R}^d) \to \mathbb{C}^N$ and $R_- : \mathbb{C}^N \to L^2(\mathbb{R}^d)$ the operators given by

$$R_+(u) = (\langle u, e_j\rangle)_{j=1\dots N}, \qquad R_- u_{-j} = \sum_{j=1}^{N} u_{-j}(Q_z + hK)e_j. \tag{4.74}$$

We study the following operator on $L^2(\mathbb{R}^d) \times \mathbb{C}^N$

$$\mathcal{P}_z = \begin{pmatrix} Q_z & R_- \\ R_+ & 0 \end{pmatrix}, \tag{4.75}$$

which is associated to the Grushin problem

$$\begin{cases} Q_z u + R_- u_- & = v \\ R_+ u & = v_+ \end{cases}, \tag{4.76}$$

with u, $v \in L^2(\mathbb{R}^d)$ and u_-, $v_+ \in \mathbb{C}^N$. Since $Q_z = Q_z + hK - hK$ with $Q_z + hK$ invertible and K compact, Q_z and then $\mathcal{P}_z$ are holomorphic families of Fredholm operators of index 0. It is therefore enough to show that $\mathcal{P}$ is injective to show that it is invertible. Assume that

$$\mathcal{P}\begin{pmatrix} u \\ u_- \end{pmatrix} = 0, \tag{4.77}$$

with $u = \sum_{j=1}^{\infty} u_j e_j$. Then, since $R_+ u = 0$, we get $u_j = 0$ for $1 \leq j \leq N$, and, since $Ke_j = 0$ for $N < j$, the equation $Q_z u + R_- u_- = 0$ becomes

$$(Q_z + hK)\Big(\sum_{j=N+1}^{\infty} u_j e_j + \sum_{j=1}^{N} u_{-j} e_j\Big) = 0. \tag{4.78}$$

Then, from (4.46), we get $u = 0$, $u_- = 0$, and $\mathcal{P}$ is invertible. We denote its inverse by

$$\mathcal{P}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}, \tag{4.79}$$

and we look for estimates on the entries of $\mathcal{P}^{-1}$. Assume (4.76) and write $u = (u', u'')$ with $u' \in \operatorname{vect}\{e_1, \dots, e_N\}$ and $u'' \in \operatorname{vect}\{e_{N+1}, \dots\}$. Since $Q_z u + R_- u_- = v$, we have

$$(Q_z + hK)\Big(u'' + \sum_{j=1}^{N} u_{-j}\Big) = v - Q_z u', \tag{4.80}$$

and

$$u'' + \sum_{j=1}^{N} u_{-j} = (Q_z + hK)^{-1} v - \big(1 - (Q_z + hK)^{-1} hK\big) u'. \tag{4.81}$$

Therefore, since $Q_z = \mathcal{O}(1)$ and using (4.46), we obtain

$$\|u''\|_{L^2} + \|u_-\|_{\mathbb{C}^N} \leq C\big(h^{-1}\|v\|_{L^2} + \|u'\|_{L^2}\big), \tag{4.82}$$

and then, since we have $\|u'\|_{L^2} = \|v_+\|_{\mathbb{C}^N}$ because $R_+ u = v_+$, we get

$$\|u\|_{L^2} + \|u_-\|_{\mathbb{C}^N} \leq C\big(h^{-1}\|v\|_{L^2} + \|v_+\|_{\mathbb{C}^N}\big). \tag{4.83}$$

Thus we have $E = \mathcal{O}(h^{-1})$, $E_- = \mathcal{O}(h^{-1})$, $E_+ = \mathcal{O}(1)$ and $E_{-+} = \mathcal{O}(1)$.

Now we follow S. H. Tang and M. Zworski [31]. For $z \in D(0, 2C_0 h)$, Q_z is invertible if and only if $E_{-+}(z)$ is invertible, and in that case,

$$Q_z^{-1} = E_+(z) E_{-+}^{-1}(z) E_-(z) - E(z), \tag{4.84}$$

which implies

$$Q_z^{-1} = \mathcal{O}(h^{-1}) \big(1 + \| E_{-+}^{-1}(z) \| \big). \tag{4.85}$$

Since $E_{-+}(z) = \mathcal{O}(1)$ as an operator on $\mathbb{C}^N$, we also have $\| E_{-+}^{-1}(z) \| = \mathcal{O}(|D(z)|^{-1})$, where $D(z) = \det E_{-+}(z)$. Now we set

$$D_w(z) = \prod_{z_j \in \Gamma(h) \cap D(0, C_0 h)} \frac{z - z_j}{h}, \tag{4.86}$$

and we know that $D(z) = D_w(z) \times G(z)$, where $G(z)$ is holomorphic. Here, we use the fact that the order of the zeros of $D(z; h)$ coincides with the multiplicity of the eigenvalues of $\widetilde{P}$. Since $\# \big(\Gamma(h) \cap D(0, C_0 h) \big)$ is uniformly bounded, we have for $z \in D(0, C_0 h)$,

$$D_w(z) = \mathcal{O}(1). \tag{4.87}$$

On the other hand, one can find $r(h) \in]C_0, 2C_0[$ such that, for z on the circle $\partial D(0, r(h))$, we have

$$|D_w(z)| \geq \varepsilon. \tag{4.88}$$

For $z \in D(0, 2C_0 h)$ we also have

$$D(z) = \mathcal{O}(1), \tag{4.89}$$

since $E_{-+} = \mathcal{O}(1)$ as an operator on $\mathbb{C}^N$. Finally if $\operatorname{Im} z > \delta h$, we have $E_{-+}^{-1}(z) = -R_+ Q_z^{-1} R_-$, thus

$$|D(z)| > \varepsilon. \tag{4.90}$$

Using (4.88) and (4.89), we obtain

$$G(z) = \mathcal{O}(1) \tag{4.91}$$

for $z \in D(0, r(h))$. Now (4.87) and (4.90) imply that

$$|G(z)| > \varepsilon, \tag{4.92}$$

for $\operatorname{Im} z > \delta h$. Then Harnack's inequality for the function $C - \ln |G(z)|$, where C is chosen so that this function is non-negative, implies

$$G(z)^{-1} = \mathcal{O}(1) \tag{4.93}$$

for $z \in D(0, C_0 h)$. Therefore, if $d(z, \Gamma(h)) > \nu h^N$, one has $\det(E_{-+}(z))^{-1} = \mathcal{O}(h^{-C})$, $(E_{-+}(z))^{-1} = \mathcal{O}(h^{-C})$ and the proposition follows from (4.85). $\square$

Finally, we extend the domain of validity of the estimate (4.45) on Q_z^{-1} as far as possible into the lower half complex plane.

Proposition 4.3. *Assume that t_2 and M^{-1} are small enough. There exists $\delta_0 > 0$ such that, for all t_1 large enough, Q_z is invertible on $L^2(\mathbb{R}^d)$ for $z \in D(0, C_0 h)$ with $\operatorname{Im} z > -\delta_0 h$, and, for such z,*

$$\| Q_z^{-1} \| = \mathcal{O}(h^{-1}). \tag{4.94}$$

Proof. From (4.48) and Fefferman-Phong's inequality, we have

$$\left\langle \operatorname{Op}_h \left(-M^{-1}\{\widetilde{g}_2, \widehat{p}_0\chi_8\}\chi_2^2\right) u, u\right\rangle \geq \varepsilon h \operatorname{Op}_{\frac{1}{M}} \left(\Big(\frac{Y^2}{\langle Y\rangle^2} + \frac{H^2}{\langle H\rangle^2}\Big)\chi_2^2(\lambda Y, \lambda H)\right) + \mathcal{O}(hM^{-2})\|u\|^2.$$

But using the Appendix of [3], we have

$$\operatorname{Op}_{\frac{1}{M}} \left(\Big(\frac{Y^2}{\langle Y\rangle^2} + \frac{H^2}{\langle H\rangle^2}\Big)\chi_2^2(\lambda Y, \lambda H)\right) \geq \varepsilon M^{-1}\chi_1^2(\lambda Y, \lambda H) + \mathcal{O}(M^{-2}), \tag{4.95}$$

and (4.47) becomes, as for (4.56),

$$\begin{aligned}\operatorname{Im}\langle Q_z u, u\rangle_{L^2(\mathbb{R}^d)} \geq& \varepsilon t_2 h M^{-1}\|\operatorname{Op}_h(\chi_1)u\|^2 + \varepsilon t_1 h\ln(1/h)\|\operatorname{Op}_h(\chi_6 - \chi_1)u\|^2\\ &+ \sqrt{h}\,\langle \operatorname{Op}_h(1-\chi_5)u, u\rangle + \operatorname{Im} z\|u\|^2\\ &- Ch(\ln(1/h) + \ln(M))\|\operatorname{Op}_h(\varphi_1)u\|^2 - Ct_1 h\ln(1/h)\|\operatorname{Op}_h(\varphi_2)u\|^2\\ &+ \left(\mathcal{O}(t_2^2 hM^{-1}) + \mathcal{O}_M(h^{1+\varepsilon}) + \mathcal{O}(hM^{-2}) + \mathcal{O}(zM^{-2})\right)\|u\|^2.\end{aligned} \tag{4.96}$$

If t_2 is fixed small enough and t_1 large enough, we obtain

$$\begin{aligned}-\operatorname{Im}\langle Q_z u, u\rangle_{L^2(\mathbb{R}^d)} \geq& \left(\varepsilon t_2 hM^{-1} + \operatorname{Im} z\right)\|u\|^2\\ &+ \left(\mathcal{O}(t_2^2 hM^{-1}) + \mathcal{O}_M(h^{1+\varepsilon}) + \mathcal{O}(hM^{-2}) + \mathcal{O}(hM^{-2})\right)\|u\|^2.\end{aligned} \tag{4.97}$$

This gives the Proposition, provided M is chosen large enough. □

Now we finish the proof of Theorem 2.1. Proposition 4.2 implies that, for $u \in L^2(\mathbb{R}^d)$ and if $d(z, \Gamma(h)) > \nu h^N$, we have

$$\|G_2^{-1}G_1^{-1}u\| = \mathcal{O}(h^{-C})\|G_{-2}G_{-1}(\widetilde{P} - z)u\|. \tag{4.98}$$

Since $G_{-2} \in \Psi_h^{1/2}(h^{-Ct_2})$, we also have $\|G_{-2}\| = \mathcal{O}(h^{-Ct_2})$. Working in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$, we get $\operatorname{Op}_h\left(e^{-t_2 g_2}\right)\operatorname{Op}_h\left(e^{t_2 g_2}\right) = 1 + \mathcal{O}(M^{-2})$, so that

$$G_2^{-1} = (1 + \mathcal{O}(M^{-2}))G_{-2} = \mathcal{O}(h^{-Ct_2}). \tag{4.99}$$

Then (4.98) becomes

$$\|G_1^{-1}u\| = \mathcal{O}(h^{-C-2Ct_2})\|G_{-1}(\widetilde{P} - z)u\|, \tag{4.100}$$

and since, from pseudodifferential calculus in $\mathcal{S}_h^0$, we have $G_1^{-1} = (1 + \mathcal{O}(h^2))G_{-1}$, this gives

$$\|G_{-1}u\| = \mathcal{O}(h^{-C-2Ct_2})\|G_{-1}(\widetilde{P} - z)u\|. \tag{4.101}$$

Now we use a Cordoba–Fefferman type estimate, as given by A. Martinez (see [17, Corollary 3.5.3]) and in a more precise form in [2, Theorem 3].

Lemma 4.4. *Let $f(y,\eta)$, $a(y,\eta,h) \in \mathcal{S}_h^0(1)$. There exist a symbol $b(y,\eta,h) \sim \sum_{j\geq 0} h^j b_j(y,\eta) \in \mathcal{S}_h^0(1)$, and an operator $R(h) = \mathcal{O}(h^\infty)$ such that, for all $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, one has*

$$\langle f\mathcal{T}'\operatorname{Op}_h(a)u, \mathcal{T}'v\rangle_{L^2(\mathbb{R}^d)} = \left\langle (b(y,\eta,h) + R(h))\mathcal{T}'u, \mathcal{T}'u\right\rangle_{L^2(\mathrm{T}^*(\mathbb{R}^d))}, \tag{4.102}$$

where $\operatorname{supp} b_j \subset \operatorname{supp} f$ *for all j, and b_j is given in terms of derivatives of a and f of order at most $2j$. In particular*

$$b_0(y,\eta) = f(y,\eta)a_0(y,\eta).$$

From this result, one can obtain the following

Corollary 4.5. *There exists a function $b(y, \eta, h) = 1 + \mathcal{O}_{t_1}(h \ln^2(1/h))$ such that*

$$\|G_{-1}u\|^2_{L^2(\mathbb{R}^d)} = \big\langle b e^{-t_1 g_1(y,\eta)} \mathcal{T}' u, e^{-t_1 g_1(y,\eta)} \mathcal{T}' u \big\rangle_{L^2(\mathrm{T}^*(\mathbb{R}^d))} + \mathcal{O}(h^\infty)\|u\|^2. \tag{4.103}$$

Then, in view of this estimate, (4.101) gives

$$\|e^{-t_1 g_1} \mathcal{T}' u\|_{L^2(\mathrm{T}^*(\mathbb{R}^d))} = \mathcal{O}(h^{-C-2Ct_2}) \|e^{-t_1 g_1} \mathcal{T}' (\widetilde{P} - z) u\|_{L^2(\mathrm{T}^*(\mathbb{R}^d))} + \mathcal{O}(h^\infty)\|u\|. \tag{4.104}$$

Now assume that u and z satisfy the assumptions of Theorem 2.1. Let $\chi_4 \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^d))$ with $\chi_3 \prec \chi_4 \prec \chi_5$ and suppose that $\mathrm{FS}((\widetilde{P} - z)u)$ does not intersect a neighborhood of the support of χ_4. Then (4.104) gives

$$\begin{aligned} \|e^{-t_1 g_1} \mathcal{T}' \operatorname{Op}_h(\chi_4) u\| =& \mathcal{O}(h^{-C-2Ct_2}) \|e^{-t_1 g_1} \mathcal{T}' \operatorname{Op}_h(\chi_4)(\widetilde{P} - z) u\| \\ &+ \mathcal{O}(h^{-C-2Ct_2}) \|e^{-t_1 g_1} \mathcal{T}' [\widetilde{P}, \operatorname{Op}_h(\chi_4)] u\| + \mathcal{O}(h^\infty)\|u\|. \end{aligned} \tag{4.105}$$

Let $\varphi_3 \in C_0^\infty(T^*(\mathbb{R}^d))$ with $\chi_4' \prec \varphi_3 \prec (1 - \chi_3)\chi_5$. Since the operator $[\widetilde{P}, \operatorname{Op}_h(\chi_4)] \in \Psi_h^0(h)$ has its symbol supported inside the support of χ_4', we get, using again Lemma 4.4,

$$\|e^{-t_1 g_1} \chi_4 \mathcal{T}' u\|_{L^2(\mathrm{T}^*(\mathbb{R}^d))} = \mathcal{O}(h^\infty) + \mathcal{O}(h^{-C-2Ct_2}) \|e^{-t_1 g_1} \varphi_3 \mathcal{T}' u\|. \tag{4.106}$$

Here the constant C is uniform with respect to t_1. Now, we can assume that the support of χ_4 and φ_3 satisfies the properties of the domains Ω_1 and $\Omega_0 \setminus \Omega_1$ as in (3.2) and Figure 2, and we get the main part of Theorem 2.1. The remaining statement concerning the fact that the exceptional set $\Gamma(h)$ can be chosen so that $\Gamma(h) \subset \{\operatorname{Im} z \leq -\delta_0 h\}$ for some $\delta_0 > 0$, follows from the above discussion, using Proposition 4.3 instead of Proposition 4.2.

5. Existence

This section is devoted to the proof of Theorem 2.5. We use the ideas and the constructions of B. Helffer and J. Sjöstrand in [14], concerning the study of the tunnel effect between potential wells. At many places in the following sections, we shall use some terminology and some general results from [14] that we recall now, here in a slightly different setting.

Let $(\mu_j)_{j \geq 0}$ be the strictly growing sequence of linear combinations over $\mathbb{N}$ of the λ_j's. Let $u(t, x, \eta')$ be a function defined on $\mathbb{R}^+ \times U \times V$, $U \subset \mathbb{R}^d$, $V \subset \mathbb{R}^m$.

Definition 5.1. *We say that $u : [0, +\infty[\times U \times V \to \mathbb{R}$, a smooth function, is expandible, if, for any $N \in \mathbb{N}$, $\varepsilon > 0$, $\alpha, \beta, \gamma \in \mathbb{N}^{1+d+m}$,*

$$\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma \Big(u(t, x, \eta') - \sum_{j=1}^N u_j(t, x, \eta') e^{-\mu_j t} \Big) = \mathcal{O}\big(e^{-(\mu_{N+1} - \varepsilon)t}\big), \tag{5.1}$$

for a sequence of $(u_j)_j$ smooth functions, which are polynomials in t. We shall write

$$u(t, x, \eta') \sim \sum_{j \geq 1} u_j(t, x, \eta') e^{-\mu_j t},$$

when (5.1) holds.

As the following result shows, this symbol class is the suitable one for our geometric setting at $(0, 0)$.

Proposition 5.2 ([14], Section 3). *Let $\nu(t,x,\eta')$ be a time-dependent vector field. Suppose that there exists a matrix-valued map $(x,\eta')\mapsto A(x,\eta')$ from $U\times V$ to $\mathcal{M}_d(\mathbb{R})$ such that*

i) $A(0)=\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_d)$, with $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_d$.
ii) $(t,x,\eta')\mapsto\nu(t,x,\eta')-A(x,\eta')x$ is a smooth real expandible matrix.

Then, if $v(t,x,\eta')$ is expandible and vanishes at $x=0$, and $u_0(x,\eta')$ is a smooth function, the solution $u(t,x,\eta')$ to the Cauchy problem

$$\begin{cases}\partial_t u+\nu(t,x,\eta')u=v, & t\ge 0,\ x\in U,\ \eta'\in V,\\ u_{|t=0}=u_0,\end{cases}\tag{5.2}$$

is expandible.

Notice that this result implies in particular that, as we have already mentioned in Section 2, $\gamma(t,x,\xi)=\exp(\pm tH_p)(x,\xi)$ is expandible when $(x,\xi)\in\Lambda_\mp$.

Definition 5.3. *We say that $u(t,x,\eta',h)$, a smooth function is of class $\mathcal{S}^{A,B}$ if, for any $\varepsilon>0$, $(\alpha,\beta,\gamma)\in\mathbb{N}^{1+d+m}$,*

$$\partial_t^\alpha\partial_x^\beta\partial_{\eta'}^\gamma u(t,x,\eta',h)=\mathcal{O}\big(h^Ae^{-(B-\varepsilon)t}\big).\tag{5.3}$$

Let $\mathcal{S}^{\infty,B}=\bigcap_A\mathcal{S}^{A,B}$. We say that $u(t,x,\eta',h)$ is a classical expandible function of order (A,B), if, for any $K\in\mathbb{N}$,

$$u(t,x,\eta',h)-\sum_{k=A}^{K}u_k(t,x,\eta')h^k\in\mathcal{S}^{K+1,B},\tag{5.4}$$

for a sequence of $(u_k)_k$ expandible functions. We shall write

$$u(t,x,\eta',h)\sim\sum_{k\ge A}u_k(t,x,\eta')h^k,$$

in that case.

We recall from Section 2 that Ω is a small neighborhood of $(0,0)\in T^*\mathbb{R}^d$, that $\varepsilon>0$ is small enough such that $S=\Lambda_-\cap\{(x,\xi);\ |x|=\varepsilon\}\subset\Omega$, and $U\subset\Omega$ a neighborhood of S. We look for a solution of the problem

$$\begin{cases}(P-z)u=0 & \text{microlocally in }\Omega,\\ u=u_0 & \text{microlocally in }U.\end{cases}\tag{5.5}$$

Since this problem is linear with respect to the initial data u_0, we can assume that u_0 vanishes microlocally outside a small neighborhood of some point $\rho_-\in(\Lambda_-\cap S)\setminus\widetilde{\Lambda_-}$. We recall that by assumption, u_0 vanishes on $\widetilde{\Lambda_-}$. Since P is of principal type in $\Omega\setminus\{(0,0)\}$, u_0 can be extended as a microlocal solution of $(P-z)u_0=0$ near each point of $\Lambda_-\setminus\{(0,0)\}$. As $\rho_-\notin\widetilde{\Lambda_-}$, we know from (2.22), that

$$\gamma_-(t)=\exp(tH_p)(\rho_-)\sim\sum_{j=1}^{+\infty}e^{-\mu_jt}\gamma_j^-(t),\ \text{as }t\to+\infty,\tag{5.6}$$

where $\gamma_1^-\neq 0$ is an eigenvector of $F_p=d_{(0,0)}H_p$ associated to the eigenvalue $-\lambda_1$. We recall that $(\mu_j)_{j\ge0}$ is the strictly growing sequence of linear combinations over $\mathbb{N}$ of the λ_j's.

Here and from now on, we shall write points in $T^*\mathbb{R}^d$ as $(x,\xi) = (x_1, x', \xi_1, \xi')$ with x_1, ξ_1 in $\mathbb{R}$ and x', ξ' in $\mathbb{R}^{d-1}$. We can always assume, up to a linear change of variables, that $g_1^-(\rho_-) = \Pi_x \gamma_1^-$ is collinear to the direction x_1. In these coordinates, we set $H_- : x_1 = \varepsilon$. Of course, the lift $H_- \times \mathbb{R}^d$ of H_- in $T^*\mathbb{R}^d$ is transverse to γ_- for ε small enough, and we can suppose so. Here and in the sequel we may have to change a certain finite number of times for a smaller $\varepsilon > 0$, and therefore to change (silently) for another ρ_- on the curve γ_-. In the rest of this section, we prove Theorem 2.5 under a more precise form. As in [14], the main idea is to look for a solution to (5.5) of the form

$$u(x,h) = \frac{1}{(2\pi h)^{d-\frac{1}{2}}} \iint_{T^*\mathbb{R}^{d-1}} \int_{-1}^{+\infty} e^{i(\varphi(t,x,\eta') - y'\eta')/h} a(t,x,\eta',z,h) u_0(\varepsilon, y') dt dy' d\eta'. \tag{5.7}$$

Therefore we shall look for a phase function φ and a symbol a such that

$$(hD_t + P(x,hD) - z) a e^{i\varphi/h} = \mathcal{O}(h^\infty), \tag{5.8}$$

in a sense that we will precise later on. One of the differences with respect to [14] is that we shall do so for each η' in a neighborhood of $\xi^{-\prime}$, so that we can also fulfill the initial condition in (5.5).

However, as in [14], in general, the integral with respect to t in (5.7) does not converge for the functions a and φ we build, and our representation of the solution is somewhat more complicated than (5.7). Recalling that we suppose $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$ for some C_0, $C_1 > 0$, we denote

$$S = S(z/h) = \sum_{j=1}^d \frac{\lambda_j}{2} - i\frac{z}{h}, \text{ and } K_1 = \mathbb{E}\Big(\frac{C_1}{\lambda_1} - \frac{\sum \lambda_j}{2\lambda_1}\Big) + 1, \tag{5.9}$$

where $\mathbb{E}(r)$ denotes the integer part of $r \in \mathbb{R}$.

Theorem 5.4. *Assume that u_0 vanishes microlocally in $\Lambda_- \cap (H_- \times \mathbb{R}^d)$ outside a small neighborhood of ρ_-. Then, there exist a neighborhood U (resp. W) of $\gamma_-([-1,+\infty[) \cup \{0\}$ (resp. $\xi^{-\prime}$) in $\mathbb{R}^d$ (resp. $\mathbb{R}^{d-1}$), a phase function $\varphi(t,x,\eta')$, a symbol $A_+(t,x,\eta',z,h)$ defined on $[-1,+\infty[\times U \times W$, and a symbol $A_-(x,\eta',z,h)$ defined on $U \times W$ such that*

i) There exists a smooth function $\tilde\psi(\eta')$ such that the function $\varphi - \varphi_+(x) - \tilde\psi(\eta')$ is expandible:

$$\varphi(t,x,\eta') - (\varphi_+(x) + \tilde\psi(\eta')) \sim \sum_{j \geq 1} e^{-\mu_j t} \varphi_j(t,x,\eta').$$

Moreover $\tilde\psi$ is a generating function for Λ_-, in the sense that, the projection of Λ_- onto T^*H_- can be written as the set of $(\nabla\tilde\psi(\eta'), \eta')$'s, with $\eta' \in W$.

ii) The symbol A_+ is classically expandible: $A_+ \in \mathcal{S}^{-K_1,-\delta}$ for some $\delta > 0$, and it is an analytic function with respect to $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$.

iii) The function A_- is a semiclassical symbol of order $-K_1$, and it is an analytic function with respect to $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$.

iv) For any cut-off function $\chi \in C^\infty(]-1,+\infty[)$ equal to 1 near $[0,+\infty[$, the function

$$
\begin{aligned}
u(x,z,h) = {}& \frac{1}{(2\pi h)^{d-\frac{1}{2}}} \iint_{T^*\mathbb{R}^{d-1}} e^{i(\varphi_+(x)+\tilde{\psi}(\eta')-y'\eta')/h} A_-(x,\eta',z,h) u_0(\varepsilon,y') dy' d\eta' \\
&+ \frac{1}{(2\pi h)^{d-\frac{1}{2}}} \iint_{T^*\mathbb{R}^{d-1}} \int_{-1}^{+\infty} e^{i(\varphi(t,x,\eta')-y'\eta')/h} \chi(t) A_+(t,x,\eta',z,h) u_0(\varepsilon,y') dt dy' d\eta',
\end{aligned} \tag{5.10}
$$

is a solution to (5.5) for any $z \in [-C_0 h, C_0 h] + i[-C_1 h, C_1 h]$.

Precise definitions for A_+ and A_- are given in Section 5.2 below. Notice that different choices for the cut-off function χ in (5.10) would lead to the same microlocal solution in Ω.

5.1. **The phase function.**

We start with the construction of the phase function φ. From (2.9), for $x' = o(x_1)$ and $\xi' = o(x_1)$, the equation $p_0(x,\xi_1,\xi') = 0$ has two solutions

$$\xi_1 = f_\pm(x,\xi') = \pm\frac{\lambda_1}{2} x_1 + o(x_1). \tag{5.11}$$

Since γ_- is a simple characteristic for the operator P, by usual Hamilton-Jacobi theory we have first the

Lemma 5.5. *There exists a neighborhood U_- of x^-, which depends on ε, such that, for all $\eta' \in \mathbb{R}^{d-1}$ close enough to $\xi^{-\prime}$, there is a unique smooth function $\psi_{\eta'} : \mathbb{R}^d \to \mathbb{R}$, defined in U_-, verifying*

$$
\begin{cases}
p_0(x,\nabla\psi_{\eta'}(x)) = 0, \\
\psi_{\eta'}(x) = x' \cdot \eta' \text{ for } x \in H_- \cap U_-, \\
\partial_{x_1}\psi_{\eta'}(x^-) = f_-(x^-,\eta').
\end{cases} \tag{5.12}
$$

If we denote by $\Lambda_{\psi_{\eta'}}$ the corresponding Lagrangian manifold

$$\Lambda_{\psi_{\eta'}} = \{(x,\xi) \in T^*\mathbb{R}^d;\ x \in U_-,\ \xi = \nabla\psi_{\eta'}(x)\}, \tag{5.13}$$

we have the following

Lemma 5.6. *The Lagrangian manifolds Λ_- and $\Lambda_{\psi_{\eta'}}$ intersect along an integral curve $\gamma_{\eta'}$ for H_p, and they intersect transversally. This curve is γ_- when $\eta' = \xi^{-\prime}$.*

Proof. First we study $\Lambda_{\psi_{\eta'}} \cap (H_- \times \mathbb{R}^d)$: a point (x_1,x',ξ_1,ξ') belongs to this intersection if and only if $x_1 = x_1^- = \varepsilon$ and $(\xi_1,\xi') = \nabla_x\psi_{\eta'}(x_1^-,x')$. But we have

$$\nabla_{x'}\psi_{\eta'}(x_1^-,x') = \eta', \tag{5.14}$$

and, moreover, $\psi_{\eta'}$ satisfies the eikonal equation. Thus, using also the third equation of (5.12), we get by continuity

$$\partial_{x_1}\psi(x_1^-,x') = f_-(x_1^-,x',\eta'), \tag{5.15}$$

and

$$\Lambda_{\psi_{\eta'}} \cap (H_- \times \mathbb{R}^d) = \big\{(x_1^-,x',f_-(x_1^-,x',\eta'),\eta'),\ x' \in \mathbb{R}^{d-1}\big\}. \tag{5.16}$$

Then the intersection of $\Lambda_{\psi_{\eta'}} \cap (H_- \times \mathbb{R}^d)$ with Λ_- is given by the equation

$$(x_1^-, x', f_-(x_1^-, x', \eta'), \eta') = (x_1^-, x', \nabla_x \varphi_-(x_1^-, x')), \tag{5.17}$$

where φ_- is a generating function for Λ_- in Ω as in (2.15).

Let $g : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ be the function defined by $g(x') = \nabla_{x'} \varphi_-(x_1^-, x')$. In view of (2.14), we have $g(x^{-\prime}) = \xi^{-\prime}$ and $\nabla_{x'} g(x^{-\prime}) = \nabla^2_{x',x'} \varphi_-(x^{-\prime}) = -L'/2 + o(1)$ as $\varepsilon \to 0$. Here L' is the $(d-1) \times (d-1)$ matrix given by $L' = \operatorname{diag}(\lambda_2, \dots, \lambda_d)$ (see (2.12)). Thus the inverse function theorem implies that $g(x') = \eta'$ has a unique solution $x' = x'(\eta')$ in a neighborhood of $x^{-\prime}$, for η' in a neighborhood of $\xi^{-\prime}$. Notice also that,

$$x'(\eta') = x^{-\prime} + \mathcal{O}(|\eta' - \xi^{-\prime}|), \tag{5.18}$$

uniformly as $\varepsilon \to 0$ in a neighborhood of $\xi^{-\prime}$ which depends on ε. Since $\Lambda_- \subset p_0^{-1}(0)$, we have

$$\partial_{x_1} \varphi_-(x_1^-, x'(\eta')) = f_-(x_1^-, x'(\eta'), \eta'), \tag{5.19}$$

so that finally the equation (5.17) has a unique solution $x'(\eta')$ in a neighborhood of $x^{-\prime}$ for η' close enough to $\xi^{-\prime}$.

Let us denote by

$$\rho_{\eta'} = (x(\eta'), \xi(\eta')) = \big(x_1^-, x'(\eta'), f_-(x_1^-, x'(\eta'), \eta'), \eta'\big) \tag{5.20}$$

the corresponding point. We show now that the tangent spaces at $\rho_{\eta'}$ to $\Lambda_{\psi_{\eta'}}$ and Λ_- intersect along a one-dimensional space.

First it is clear that H_p belongs to both $T_{\rho_{\eta'}} \Lambda_{\psi_{\eta'}}$ and $T_{\rho_{\eta'}} \Lambda_-$, since Λ_- as well as $\Lambda_{\psi_{\eta'}}$ are invariant under the H_p flow, or otherwise stated, because these Lagrangian manifolds are generated by solutions of the eikonal equation for p.

On the other hand, a vector (δ_x, δ_ξ) belongs to $T_{\rho_{\eta'}} \Lambda_{\psi_{\eta'}} \cap T_{\rho_{\eta'}} \Lambda_-$ if and only if

$$\left\{ \begin{aligned} &\delta_\xi = (\nabla^2_{x,x} \psi_\eta)(x(\eta')) \delta_x, \\ &\delta_\xi = (\nabla^2_{x,x} \varphi_-)(x(\eta')) \delta_x, \end{aligned} \right. \tag{5.21}$$

or $\delta_x \in \operatorname{Ker} \big((\nabla^2_{x,x} \psi_{\eta'})(x(\eta')) - (\nabla^2_{x,x} \varphi_-)(x(\eta')) \big)$. But we have seen that $\nabla^2_{x,x} \varphi_-(\rho_-) = -L/2 + o(1)$ as $\varepsilon \to 0$, and that $(\nabla^2_{x',x'} \psi_{\eta'})(\rho_x(\eta')) = 0$, so that the matrix $(\nabla^2_{x,x} \psi_{\eta'})(x(\eta')) - (\nabla^2_{x,x} \varphi_-)(x(\eta'))$ has a $(d-1) \times (d-1)$ non-vanishing minor. Thus its rank is larger than $d-1$, and finally H_p generates $T_{\rho_{\eta'}} \Lambda_{\psi_{\eta'}} \cap T_{\rho_{\eta'}} \Lambda_-$. $\square$

Let $\gamma_{\eta'}$ be the hamiltonian curve with initial data $\rho(\eta')$. We denote by $\Gamma_0^{\eta'}$ the set of level $\psi_{\eta'}(x(\eta'))$ for $\psi_{\eta'}$:

$$\Gamma_0^{\eta'} = \{(x, \xi) \in \Lambda_{\psi_{\eta'}};\ \psi_{\eta'}(x) = \psi_{\eta'}(x(\eta'))\}, \tag{5.22}$$

and, possibly after shrinking U_-, we have the

Lemma 5.7. *For ε small enough, there exists a neighborhood V_- of $\xi^{-\prime}$ such that, for any $\eta' \in V_-$, one can find a Lagrangian manifold $\Lambda_0^{\eta'}$ defined above U_- such that*

$$\Lambda_0^{\eta'} \cap \Lambda_{\psi_{\eta'}} = \Gamma_0^{\eta'}, \tag{5.23}$$

where the intersection is clean. Moreover $\Lambda_0^{\eta'}$ *depends smoothly on* η'*, and* $\Pi_x : \Lambda_0^{\eta'} \to U_-$ *is a diffeomorphism.*

Proof. It is sufficient to prove the Lemma for $\eta' = \xi^{-\prime}$. Indeed, every object that appears below evaluated at $\xi^{-\prime}$ is a smooth functions of $\eta' \in V_-$. In particular the estimates below hold uniformly with respect to η'.

The vector (δ_x, δ_ξ) belongs to $T_{\rho_-}\Gamma_0^{\xi^{-\prime}}$ if and only if

$$\begin{cases} \delta_\xi = (\nabla^2_{x,x}\psi_{\xi^{-\prime}})(x^-)\delta_x, \\ (\nabla_x \psi_{\xi^{-\prime}})(x^-)\delta_x = 0. \end{cases} \tag{5.24}$$

Indeed $\Gamma_0^{\xi^{-\prime}} \subset \Lambda_\psi^{\xi^{-\prime}}$, and $\Gamma_0^{\xi^{-\prime}}$ is a level curve for ψ. Thanks to (5.14) and (5.15), the second equation becomes

$$\delta_x^1 = -\frac{\xi^{-\prime} \cdot \delta_x'}{f_-(x^-, \xi^{-\prime})} \tag{5.25}$$

and we see in particular that $T_{\rho_-}\Gamma_0^{\xi^{-\prime}}$ is parametrized by δ_x'.

Let us compute the entries of the matrix $M_\varepsilon = (\nabla^2_{x,x}\psi_{\xi^{-\prime}})(x^-)$. We have already seen (see (5.14)) that, for $i, j \geq 2$, $m_{ij} = 0$. We also know (see (5.15)) that

$$(\nabla_{x'}\partial_{x_1}\psi)(x^-) = \nabla_{x'} f_-(x^-, \xi^{-\prime}) = \frac{L'^2 x^{-\prime} + \mathcal{O}(|x_1^-|^2)}{4f(x^-, \xi^{-\prime})}, \tag{5.26}$$

and we are left with the computation of $\partial^2_{x_1,x_1}\psi(x^-)$. But we have seen that $H_p(\rho_-) = (\nabla_\xi p_0(\rho_-), -\nabla_x p_0(\rho_-))$ belongs to $T_{\rho_-}\Lambda_\psi^{\xi^{-\prime}}$, that is satisfies the first equation in (5.24), so that

$$\frac{1}{2}L^2 x^- + \mathcal{O}(|x_1^-|^2) = M_\varepsilon\big(2\xi^- + \mathcal{O}(|x_1^-|^2)\big) \tag{5.27}$$

which gives in particular

$$\frac{\lambda_1^2}{2}x_1^- + \mathcal{O}(|x_1^-|^2) = m_{11}(2\xi_1^- + \mathcal{O}(|x_1^-|^2)) + \frac{L'^2 x^{-\prime} \cdot \xi^{-\prime} + \mathcal{O}(|x_1^-|^3)}{2\xi_1^-}, \tag{5.28}$$

so that

$$m_{11} = -\frac{\lambda_1}{2} + o(1). \tag{5.29}$$

as $\varepsilon \to 0$. Here we recall that, as $x_1^- = \varepsilon$ goes to 0, we have $\xi_1^- = -\frac{\lambda_1}{2}\varepsilon + o(\varepsilon)$, $x^{-\prime} = o(\varepsilon)$ and $\xi^{-\prime} = o(\varepsilon)$.

Thus

$$M_\varepsilon = \begin{pmatrix} -\frac{\lambda_1}{2} & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} + o(1), \tag{5.30}$$

and the equation (5.25) becomes

$$\delta_x^1 = o(\delta_x'). \tag{5.31}$$

Summing up, we see, using (5.24), that the vectors of $T_{\rho_-}\Gamma_0^{\xi^{-\prime}}$ can be written, when $\varepsilon \to 0$, as

$$\big((0,\delta_x'),(0,0)\big) + o(\delta_x'),\ \ \delta_x' \in \mathbb{R}^{d-1}. \tag{5.32}$$

Let us denote by $\mathcal{E}_0$ the "limit space" for $T_{\rho_-}\Gamma_0^{\xi^{-\prime}}$, that is the linear subspace of $\mathbb{R}^{2d}$ generated by the e_j's for $j = 2,\dots,d$, where $e_j = (\delta_{i,j})_{i=1,\dots,2d}$. It is clear that $e_1\mathbb{R} \oplus \mathcal{E}_0$ is a Lagrangian subspace of $\mathbb{R}^{2d}$. Then, using the Gram–Schmidt orthonormalization principle, one can find a unitary vector $v(\varepsilon) \in \mathbb{R}^{2d}$ such that

$$\sigma\left(u, v(\varepsilon)\right) = 0,\ \text{ for all } u \in T_{\rho_-}\Gamma_0^{\xi^{-\prime}}, \tag{5.33}$$

and

$$v(\varepsilon) = e_1 + o(1), \tag{5.34}$$

as $\varepsilon \to 0$. Then, $v(\varepsilon)\mathbb{R} \oplus T_{\rho_-}\Gamma_0^{\xi^{-\prime}}$ is a Lagrangian vector space at ρ_-, and, extending $\Gamma_0^{\xi^{-\prime}}$ along a suitably chosen Hamilton field, one can find locally close to ρ_-, a Lagrangian manifold $\Lambda_0^{\xi^{-\prime}}$ such that $\Gamma_0^{\xi^{-\prime}} \subset \Lambda_0^{\xi^{-\prime}}$ and

$$T_{\rho_-}\Lambda_0^{\xi^{-\prime}} = v(\varepsilon)\mathbb{R} \oplus T_{\rho_-}\Gamma_0^{\xi^{-\prime}}. \tag{5.35}$$

Moreover, if $(\delta_x,\delta_\xi) \in T_{\rho_-}\Lambda_0^{\xi^{-\prime}}$, we get, from (5.32) and (5.34),

$$\delta_\xi = o(\delta_x). \tag{5.36}$$

To show that the intersection $\Lambda_0^{\xi^{-\prime}} \cap \Lambda_\psi^{\xi^{-\prime}}$ is clean, is is enough to show that $H_p(\rho_-) \in T_{\rho_-}\Lambda_\psi^{\xi^{-\prime}}$ is not in $T_{\rho_-}\Lambda_0^{\xi^{-\prime}}$. As $\varepsilon \to 0$, we have

$$H_p(\rho_-) = \begin{pmatrix} 2\xi_1^- \\ 2\xi^{-\prime} \\ \lambda_1 x_1^-/2 \\ L'x^{-\prime}/2 \end{pmatrix} + \mathcal{O}(\varepsilon^2) = \begin{pmatrix} -\lambda_1\varepsilon/2 \\ 0 \\ \lambda_1\varepsilon/2 \\ 0 \end{pmatrix} + o(\varepsilon). \tag{5.37}$$

Then (5.36) implies that $H_p(\rho_-) \notin T_{\rho_-}\Lambda_0^{\xi^{-\prime}}$ and the dimension of the intersection $T_{\rho_-}\Lambda_0^{\xi^{-\prime}} \cap T_{\rho_-}\Lambda_\psi^{\xi^{-\prime}}$ is exactly $d-1$.

Finally, the Lagrangian manifold $\Lambda_0^{\xi^{-\prime}}$ projects nicely on the x-space: Indeed if $(\delta_x,\delta_\xi) \in T_{\rho_-}\Lambda_0^{\xi^{-\prime}}$, we know by (5.36), that as $\varepsilon \to 0$, $\delta_\xi = o(\delta_x)$, so that $\delta_x \neq 0$ for any ε small enough. $\square$

Now we consider the associated Lagrangian manifold

$$\Lambda_t^{\eta'} = \exp(tH_p)(\Lambda_0^{\eta'}). \tag{5.38}$$

The manifold $\Lambda_t^{\eta'}$ projects nicely on $\mathbb{R}_x^d$. In fact, possibly after shrinking V_-, we have the

Lemma 5.8. *There exists $T_0 > 0$ such that for any $\varepsilon > 0$ small enough, there exist $\delta > 0$ and V_- a neighborhood of $\xi^{-\prime}$ such that for all $\eta' \in V_-$, the manifold $\Lambda_t^{\eta'}$ projects nicely onto $U_t = B(x_-(t),\delta)$ for $t \in [-1,T_0]$ and onto U_∞ for $t > T_0$. Here U_∞ is a neighborhood of $0 \in \mathbb{R}^d$, such that $B(x_-(t),\delta) \subset U_\infty$, for $t > T_0$.*

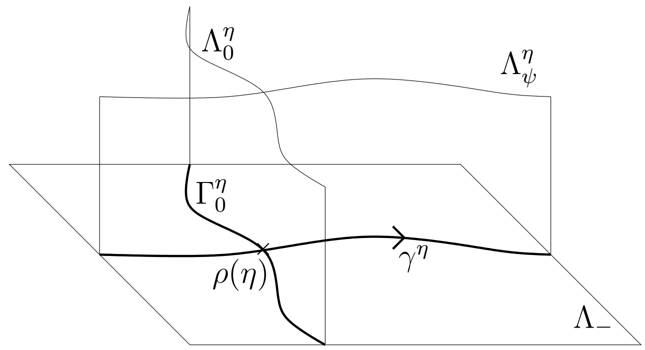

FIGURE 3. The Lagrangian manifolds.

Proof. Let (δ_x, δ_ξ) be in the tangent space $T_{\rho_{\eta'}(t)}\Lambda_t^{\eta'}$. The proof of Lemma 2.1 of [14] implies that

$$\delta_\xi - L/2\delta_x = B_t\left(\delta_\xi + L/2\delta_x\right), \tag{5.39}$$

with $B_t = \mathcal{O}\left(e^{-\lambda_1 t}\right)$, uniformly with respect to ε and η'. Then, for each $\widetilde{\varepsilon} > 0$, there is a $T_0 > 0$, such that

$$|\delta_\xi - L/2\delta_x| \leq \widetilde{\varepsilon}|\delta_x|, \tag{5.40}$$

for $(\delta_x, \delta_\xi) \in T_{\rho_{\eta'}(t)}\Lambda_t^{\eta'}$, $t > T_0$, uniformly with respect to ε and η'. This inequality, together with [14, Lemma 2.2], gives the Proposition for $t > T_0$.

For $t \in [-1, T_0]$, it is enough to prove the Lemma for $\eta' = \xi^{-\prime}$, as in the proof of Lemma 5.7. We shall use the fact that, on $[-1, T_0]$, the evolution of a tangent vector is closed to the evolution for the reference operator $p_0 = \xi^2 - \sum \lambda_j^2 x_j^2/4$, provided ε is small enough. If $(\delta_x(t), \delta_\xi(t)) \in T_{\rho_-(t)}\Lambda_t^{\xi^{-\prime}}$ is the evolution of a tangent vector (δ_x, δ_ξ) along the integral curve γ_-, we have

$$\begin{aligned} \delta_x^j(t) =& \frac{1}{2}(e^{\lambda_j t} + e^{-\lambda_j t})\delta_x^j + \frac{1}{\lambda_j}(e^{\lambda_j t} - e^{-\lambda_j t})\delta_\xi^j + o(\delta_x) \\ =& \frac{1}{2}(e^{\lambda_j t} + e^{-\lambda_j t})\delta_x^j + o(\delta_x), \end{aligned} \tag{5.41}$$

$$\begin{aligned} \delta_\xi^j(t) =& \frac{\lambda_j}{4}(e^{\lambda_j t} - e^{-\lambda_j t})\delta_x^j + \frac{1}{2}(e^{\lambda_j t} + e^{-\lambda_j t})\delta_\xi^j + o(\delta_x) \\ =& \frac{\lambda_j}{4}(e^{\lambda_j t} - e^{-\lambda_j t})\delta_x^j + o(\delta_x), \end{aligned} \tag{5.42}$$

since $\delta_\xi = o(\delta_x)$ by (5.36). From (5.41) and (5.42), we see that $\delta_\xi(t)$ is a function of $\delta_x(t)$, and that proves the Lemma. $\square$

We set

$$\widetilde{U}_t = \begin{cases} U_t & \text{for } t \in [-1, T_0], \\ U_\infty & \text{for } t \in]T_0, +\infty[. \end{cases} \tag{5.43}$$

Thanks to Lemma 5.8, there is a smooth function $\varphi(t,x,\eta')$ defined on $]-1,+\infty[\times\widetilde{U}_t\times V_-$ such that the Lagrangian manifold $\Lambda_t^{\eta'}$ is given by

$$\xi=\nabla_x\varphi(t,x,\eta')\quad\text{for } x\in\widetilde{U}_t. \tag{5.44}$$

It satisfies of course the eikonal equation

$$\partial_t\varphi(t,x,\eta')+p_0(x,\nabla_x\varphi(t,x,\eta'))=0. \tag{5.45}$$

Therefore, it follows from [14, Theorem 3.12], that $\varphi(t,x,\eta')$ is expandible in the sense of Definition 5.1: There exists a sequence φ_j of smooth functions on $]-1,+\infty[\times\widetilde{U}_t\times V_-$ which are polynomials in t, such that for any $N,k\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}^d$,

$$\partial_t^k\partial_x^\alpha\partial_{\eta'}^\beta\Big(\varphi(t,x,\eta')-\sum_{j=0}^N\varphi_j(t,x,\eta')e^{-\mu_jt}\Big)=\mathcal{O}(e^{-\mu_Nt}). \tag{5.46}$$

Now we set

$$\Gamma_t^{\eta'}=\exp(tH_p)\Gamma_0^{\eta'}, \tag{5.47}$$

and we have, possibly after shrinking $\widetilde{U}_t$ and V_-, the

Proposition 5.9. *For each $\eta'\in V_-$ and $x\in\bigcup_t\widetilde{U}_t\cap\{x;\ 0<|x'|<\nu x_1\}$, for some $\nu>0$, there is a unique time $t=t(x,\eta')$ such that $x\in\Pi_x\Gamma_t^{\eta'}$. Moreover, it is the only critical point for the function $t\mapsto\varphi(t,x,\eta')$, and it is a non-degenerate critical point.*

Proof. If $x\in\Pi_x\Gamma_t^{\eta'}$, there is a $\xi\in\mathbb{R}^n$ such that $(x,\xi)\in\Gamma_t^{\eta'}$. Then

$$\xi=\nabla_x\varphi(t,x,\eta'), \tag{5.48}$$

and $p_0(x,\xi)=0$ since $\Gamma_0^{\eta'}\subset p_0^{-1}(0)$ and the Hamiltonian flow preserves the energy. Together with (5.45), we get that t is a critical point for the function $t\mapsto\varphi(t,x,\eta')$ if and only if $x\in\Pi_x\Gamma_t^{\eta'}$.

In the case $x\in U_\infty$, the proposition follows from [14, Lemma 3.14]. For $x\notin U_\infty$, it is enough to see that $\partial_t^2\varphi(t,x,\xi^{-\prime})>0$ for $x=x^-(t)$. The eikonal equation (5.45) implies

$$\nabla_x\partial_t\varphi=-2\operatorname{Hess}(\varphi)\nabla_x\varphi+L^2x/2+\mathcal{O}\big((x^2+|\nabla_x\varphi|^2)(|\operatorname{Hess}(\varphi)\nabla_x\varphi|+1)\big) \tag{5.49}$$

$$\partial_t^2\varphi=-2\nabla_x\partial_t\varphi\cdot\nabla_x\varphi+\mathcal{O}\big(|\nabla_x\partial_t\varphi|(x^2+|\nabla_x\varphi|^2)\big). \tag{5.50}$$

From (5.36), (5.40), (5.41) and (5.42), we get that $\operatorname{Hess}(\varphi)=\mathcal{O}(1)$, and that

$$\operatorname{Hess}(\varphi)>o(1), \tag{5.51}$$

as $\varepsilon\to0$ because $\delta_\xi=\operatorname{Hess}(\varphi)\delta_x$ for $(\delta_x,\delta_\xi)\in T_{\rho_-(t)}\Lambda_t^{\xi^{-\prime}}$. Since we assume that $\Pi_x\gamma_1$ is collinear to x_1 (see the remark after (5.6)), we also have

$$\begin{aligned}x^-(t)&=(x_1^-(t),0,\dots,0)+o(x_1^-(t))\\ \xi^-(t)&=(-\lambda_1x_1^-(t)/2,0,\dots,0)+o(x_1^-(t)).\end{aligned}$$

and then (5.49) and (5.50) become

$$
\begin{aligned}
\nabla_x \partial_t \varphi(t, x^-(t), \xi^{-\prime}) &= -2\mathrm{Hess}(\varphi)\xi^-(t) + L^2 x^-(t)/2 + \mathcal{O}\big(x^-(t)^2\big) && (5.52)\\
\partial_t^2 \varphi(t, x^-(t), \xi^{-\prime}) &= -2\nabla_x \partial_t \varphi \cdot \xi^-(t) + \mathcal{O}\big(x^-(t)^3\big)\\
&= 4{}^t\xi^-(t)\mathrm{Hess}(\varphi)\xi^-(t) - L^2 x^-(t)\cdot \xi^-(t) + \mathcal{O}\big(x^-(t)^3\big)\\
&\geq -L^2 x^-(t)\cdot\xi^-(t) + o\big(x^-(t)^2\big)\\
&\geq \lambda_1^3 (x^-(t))^2 + o\big(x^-(t)^2\big)\\
&> 0. && (5.53)
\end{aligned}
$$

□

As a consequence of Proposition 5.9, we get in particular that, in

$$
\widehat{U} = \bigcup_t \widetilde{U}_t \cap \{x;\ 0 < |x'| < \nu x_1\} \tag{5.54}
$$

where both these functions are defined, we have

$$
\nabla_x \psi_{\eta'}(x) = \nabla_x \left(\varphi(t(x, \eta'), x, \eta')\right). \tag{5.55}
$$

Therefore $x \mapsto \psi_{\eta'}(x)$ and $x \mapsto \varphi(t(x,\eta'), x, \eta')$ differ from a constant. Then, adding a constant (with respect to t, x) to $\varphi(t, x, \eta')$, we can assume that

$$
\varphi(t(x, \eta'), x, \eta') = x' \cdot \eta', \tag{5.56}
$$

for any $x \in H_- \cap \widehat{U}$. Furthermore, we can compute the first term in the expansion (5.46):

Lemma 5.10. *In the sense of expandible functions, we have*

$$
\varphi(t, x, \eta) \sim \varphi_+(x) + \widetilde{\psi}(\eta') + \sum_{j\geq 1} e^{-\mu_j t}\varphi_j(t, x, \eta'), \tag{5.57}
$$

where the $\varphi_j(t, x, \eta')$ are polynomials in t with smooth coefficients in x, η', and

$$
\widetilde{\psi}(\eta') = x'(\eta') \cdot \eta' - \varphi_-(x(\eta')). \tag{5.58}
$$

Proof. As we have already mentioned, the asymptotic (5.57) follows from the proofs of sections 2 and 3 of [14], and we are left with the proof of (5.58). Let us denote by $(x(t), \xi(t))$ the points on the curve $\gamma_{\eta'}$ defined in Lemma 5.6, with $(x(0), \xi(0)) = \rho_{\eta'} = (x(\eta'), \xi(\eta')) \in H_- \times \mathbb{R}^d$ given by (5.20). We notice that, by (5.57),

$$
\widetilde{\psi}(\eta') = \lim_{t\to+\infty} \varphi(t, x(t), \eta'). \tag{5.59}
$$

On the other hand, by the eikonal equation (5.45) and since $(x(t), \xi(t)) \in \gamma_{\eta'} \subset p_0^{-1}(0)$, we have

$$
\begin{aligned}
\partial_t(\varphi(t, x(t), \eta')) &= (\partial_t\varphi)(t, x(t), \eta') + (\partial_x\varphi)(t, x(t), \eta')\cdot(\partial_t x)(t)\\
&= \xi(t)\cdot(\partial_t x)(t) = \nabla\varphi_-(x(t))\cdot(\partial_t x)(t)\\
&= \partial_t(\varphi_-(x(t))) && (5.60)
\end{aligned}
$$

where we use also the fact that $\gamma_{\eta'} \subset \Lambda_-$. Therefore, we get, with (5.56),

$$\begin{aligned}\widetilde{\psi}(\eta') &= \lim_{t\to+\infty} \varphi(t, x(t), \eta') - \varphi_-(x(t)) \\ &= \varphi(0, x(0), \eta') - \varphi_-(x(0)) = x'(\eta') \cdot \eta' - \varphi_-(x(\eta')),\end{aligned} \tag{5.61}$$

which is (5.58). $\square$

5.2. **The symbol.**

Now we look for a symbol $a(t, x, \eta', z, h) = \sum_k a_k(t, x, \eta', z)h^k$ such that (5.8) holds. This leads to the usual transport equations for the a_j's (see [23, Theorem IV-19]):

$$\begin{cases} \partial_t a_0 + \partial_\xi p_0(x, \partial_x\varphi)\partial_x a_0 + \Big(\dfrac{1}{2} \operatorname{tr}\big(\partial^2_{\xi,\xi} p_0(x, \partial_x\varphi)\partial^2_{x,x}\varphi\big) - i\dfrac{z}{h}\Big) a_0 = 0, \\ \partial_t a_k + \partial_\xi p_0(x, \partial_x\varphi)\partial_x a_k + \Big(\dfrac{1}{2} \operatorname{tr}\big(\partial^2_{\xi,\xi} p_0(x, \partial_x\varphi)\partial^2_{x,x}\varphi\big) - i\dfrac{z}{h}\Big) a_k = F_k, \qquad k \geq 1, \end{cases} \tag{5.62}$$

where $F_k(a_0, \dots, a_{k-1})$ is a differential operator on the $a_0, \dots, a_{k-1}$ with smooth coefficients. In the Schrödinger case ($p = \xi^2 + V(x)$), these equations become the more familiar

$$\begin{cases} \partial_t a_0 + 2\nabla_x\varphi \cdot \nabla_x a_0 + (\Delta_x\varphi \ - i\dfrac{z}{h})a_0 = 0, \\ \partial_t a_k + 2\nabla_x\varphi \cdot \nabla_x a_k + (\Delta_x\varphi \ - i\dfrac{z}{h})a_k = i\Delta_x a_{k-1}, \qquad k \geq 1. \end{cases} \tag{5.63}$$

Let us denote by $x_{\eta'}(t)$ the spacial projection of the curve $\gamma_{\eta'}$ defined in Lemma 5.6. As in [14], using the time-dependent change of coordinates $y = x - x_{\eta'}(t)$, the transport equations (5.62) can be written as

$$\begin{aligned}\partial_t a_k + \Big(\partial_\xi p_0\big(x_{\eta'}(t) + y, \partial_x\varphi(t, x_{\eta'}(t) + y)\big) - \partial_\xi p_0\big(x_{\eta'}(t), \partial_x\varphi(t, x_{\eta'}(t))\big)\Big)\partial_y a_k \\ + \Big(\frac{1}{2} \operatorname{tr}\big(\partial^2_{\xi,\xi} p_0(\cdot, \partial_x\varphi(t, \cdot))\partial^2_{x,x}\varphi(t, \cdot)\big) - i\frac{z}{h}\Big)(x_{\eta'}(t) + y)a_k = F_k.\end{aligned} \tag{5.64}$$

We also want that the function u given by (5.7) satisfies the initial condition $u = u_0$ microlocally in U. Performing a formal stationary phase expansion with respect to t in (5.7), we get, for $x = (\varepsilon, x') \in H_-$,

$$u(x, h) = \frac{1}{(2\pi h)^{d-1}} \iint_{T^*\mathbb{R}^{d-1}} e^{i(x'\cdot\eta' - y'\cdot\eta')/h} \widetilde{a}(x', \eta', z, h)u_0(\varepsilon, y')dy'd\eta', \tag{5.65}$$

where $\widetilde{a}(x', \eta', z, h)$ is another classical symbol, whose principal part is given by

$$\widetilde{a}_0 = e^{i\pi/4} \frac{a_0(t(x, \eta'), x, \eta', z, h)}{|\partial^2_{tt}\varphi(t(x, \eta'), x, \eta')|^{1/2}}. \tag{5.66}$$

Since we want that $u(x, h)$ coincides with $u_0(x, h)$ on H_-, we look for a symbol $a(t, x, \eta', z, h)$ such that

$$\widetilde{a}(x', \eta', z, h) = 1 + \mathcal{O}(h^\infty). \tag{5.67}$$

From the structure of the stationary phase expansion, there exists a unique formal classical symbol $a_{ini}(x', \eta', z, h)$ which solves the problem (5.67). And since the vector field $(\partial_t, \nabla_x\varphi \cdot \nabla_x)$ is not tangent to the hypersurface $\mathbb{R} \times H_-$ in $\mathbb{R}^{d+1}$, we can determine uniquely solutions a_j to the problem (5.62) which satisfy

$$a(t, x, \eta', z, h) = a_{ini}(x', \eta', z, h), \tag{5.68}$$

for all $x = (\varepsilon, x') \in H_-$ and $t \in \mathbb{R}$. Notice that the a_j's depend holomorphically on the parameter z.

Moreover, by (5.64) and Proposition 5.2, the functions a_k are expandible with respect to (x, η') in the modified sense that the family of exponents is now $(S + \mu_j)_{j\in\mathbb{N}}$, where

$$S = S(z/h) = \Delta\varphi_+(0) - i\frac{z}{h} = \sum_{j=1}^{d} \frac{\lambda_j}{2} - i\frac{z}{h}. \tag{5.69}$$

We can also find a realization a, holomorphic with respect to z, of the asymptotic sum $\sum a_k(t, x, \eta', z)h^k$ such that

$$a(t, x, \eta', z, h) \in \mathcal{S}^{0,\operatorname{Re} S} \tag{5.70}$$

and

$$r = e^{-i\varphi/h}(hD_t + P(x, hD) - z)ae^{i\varphi/h} \in \mathcal{S}^{\infty,\operatorname{Re} S}. \tag{5.71}$$

Now we want to give a meaning to the integral

$$\int_{-1}^{+\infty} e^{i\varphi(t,x,\eta')/h}\chi(t)a(t, x, \eta', z, h)dt, \tag{5.72}$$

where $\chi \in C^\infty(]-1, +\infty[)$ equal to 1 near $[0, +\infty[$. Notice that with respect to the situation in [14, Section 4], here we have to deal with an oscillatory integral. As soon as $\operatorname{Re} S > 0$, this integral is absolutely convergent. But if $\operatorname{Re} S \leq 0$, there might exist j's in $\mathbb{N}$ such that $\operatorname{Re} S + \mu_j \leq 0$, and then the integral above has no obvious meaning. Nevertheless, we explain now how to obtain a solution even in that case.

We set

$$K_1 = \mathbb{E}\Big(\frac{C_1}{\lambda_1} - \frac{\sum \lambda_j}{2\lambda_1}\Big) + 1, \tag{5.73}$$

and we also denote $\varphi_\infty(x, \eta') = \varphi_+(x) + \widetilde{\psi}(\eta')$, $\varphi_\star(t, x, \eta') = \varphi - \varphi_\infty = \mathcal{O}(e^{-\lambda_1 t})$. Then we can write

$$ae^{i\varphi/h} = ae^{i\varphi/h} - \sum_{k<K_1} \frac{a}{k!}\Big(\frac{i\varphi}{h}\Big)^k e^{i\varphi_\infty/h} + \sum_{k<K_1} \frac{a}{k!}\Big(\frac{i\varphi}{h}\Big)^k e^{i\varphi_\infty/h}. \tag{5.74}$$

From our choice for K_1, there exists $\delta > 0$ such that for all $(\alpha, \beta, \gamma) \in \mathbb{N}^{1+d+(d-1)}$ and $z \in [-C_0h, C_0h] + i[-C_1h, C_1h]$,

$$\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma \Big(ae^{i\varphi/h} - \sum_{k<K_1} \frac{a}{k!}\Big(\frac{i\varphi_\star}{h}\Big)^k e^{i\varphi_\infty/h}\Big) \lesssim h^{-K_1-|\alpha|-|\beta|-|\gamma|}e^{-3\delta t}, \tag{5.75}$$

uniformly with respect to h and t.

On the other hand,

$$b = \sum_{k<K_1} \frac{a}{k!}\Big(\frac{i\varphi_\star}{h}\Big)^k \sim \sum_{1-K_1\leq k} b_k(t, x, \eta', z)h^k \tag{5.76}$$

is expandible for the family of exponents $(S + \mu_j)_j$:

$$b_k(t, x, \eta', z) \sim \sum_j b_{k,j}(t, x, \eta', z)e^{-(S+\mu_j)t}, \tag{5.77}$$

where $b_{k,j}$ is polynomial with respect to t. Let $J_1 \in \mathbb{N}$ be such that

$$\mu_{J_1} > 2\delta - \sum_j \lambda_j/2 + C_1. \tag{5.78}$$

As in [14], for an expandible symbol satisfying (5.76) and (5.77), we define

$$[b_k]_- = \sum_{j<J_1} b_{k,j} e^{-(S+\mu_j)t} \in \mathcal{S}^{0,\operatorname{Re} S}, \text{ and } [b_k]_+ = b_k - [b_k]_- \in \mathcal{S}^{0,2\delta}. \tag{5.79}$$

Using Borel's lemma, we can find $[b]_+$ and then $[b]_-$, holomorphic with respect to z in $[-C_0h, C_0h] + i[-C_1h, C_1h]$, such that

$$[b]_+ \sim \sum_{-K_1\leq k} [b_k]_+ h^k \in \mathcal{S}^{1-K_1,2\delta}, \text{ and } [b]_- = b - [b]_+ \in \mathcal{S}^{1-K_1,\operatorname{Re} S}. \tag{5.80}$$

Then the function

$$A_+(t,x,\eta',z,h) = ae^{i\varphi/h} - \sum_{k<K_1} \frac{a}{k!}\Big(\frac{i\varphi_\star}{h}\Big)^k e^{i\varphi_\infty/h} + [b]_+ e^{i\varphi_\infty/h}, \tag{5.81}$$

satisfies an estimate like (5.75), with δ instead of 3δ. As in [14, Lemma 4.1], A_+ satisfies

Proposition 5.11. *For all $(\alpha,\beta,\gamma) \in \mathbb{N}^{1+d+(d-1)}$ and $N > 0$, we have, uniformly with respect to $z \in [-C_0h, C_0h] + i[-C_1h, C_1h]$,*

$$\big|\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma (hD_t + P(x,hD) - z)A_+\big| \leq C_{\alpha,\beta,N} h^N e^{-\delta t} \tag{5.82}$$

Proof. The main difference with [14, Lemma 4.1] is that, here, P is a pseudodifferential operator. Let $c(t,x,\eta',z,h)$ be an expandible symbol like b (see (5.77)). From the definition of $[c_k]_+$ given by (5.79), we have

$$\partial_t[c_k]_+ = [\partial_t c_k]_+ \quad \text{and} \quad \partial_{\eta'}[c_k]_+ = [\partial_{\eta'} c_k]_+, \tag{5.83}$$

so that

$$\partial_t[c]_+ - [\partial_t c]_+ \in \mathcal{S}^{\infty,2\delta} \text{ and } \partial_{\eta'}[c]_+ - [\partial_{\eta'} c]_+ \in \mathcal{S}^{\infty,2\delta}. \tag{5.84}$$

Let Q be a pseudodifferential operator with classical symbol $q(x,\eta',\xi,z,h) \in \mathcal{S}_h^0(1)$ that doesn't depend on t. Then, there exist $(Q_{\widetilde{k}})_{\widetilde{k}\in\mathbb{N}}$, a family of differential operators in x with $S^0(1)$ coefficients, such that, for all $d(t,x,\eta',z,h) \in \mathcal{S}^{A,B}$ with $A, B > 0$,

$$Qd = \sum_{\widetilde{k}\geq 0} (Q_{\widetilde{k}} d) h^{\widetilde{k}} \quad \text{modulo } \mathcal{S}^{\infty,B}. \tag{5.85}$$

Moreover, if d is a classical expandible symbol, Qd is also a classical expandible symbol.

Using this property with the c_k's, we get

$$\begin{aligned} Qc &\sim \sum_{k\geq 0} Qc_k h^k \quad \text{modulo } \mathcal{S}^{\infty,\operatorname{Re} S} \\ &\sim \sum_{l\geq 0} \Big(\sum_{k+\widetilde{k}=l} Q_{\widetilde{k}} c_k \Big) h^l \quad \text{modulo } \mathcal{S}^{\infty,\operatorname{Re} S}. \end{aligned} \tag{5.86}$$

Since Q doesn't depend on t, we have

$$[Q_{\widetilde{k}} c_k]_+ = Q_{\widetilde{k}} [c_k]_+. \tag{5.87}$$

Then, (5.86) and (5.87) imply that Qc is a classical expandible symbol and

$$\begin{aligned}
[Qc]_+ \sim & \sum_{l\geq 0}\Big[\Big(\sum_{k+\widetilde{k}=l} Q_{\widetilde{k}} c_k\Big)\Big]_+ h^l \quad \text{modulo } \mathcal{S}^{\infty,2\delta} \\
\sim & \sum_{l\geq 0}\Big(\sum_{k+\widetilde{k}=l} [Q_{\widetilde{k}} c_k]_+\Big) h^l \quad \text{modulo } \mathcal{S}^{\infty,2\delta} \\
\sim & \sum_{l\geq 0}\Big(\sum_{k+\widetilde{k}=l} Q_{\widetilde{k}} [c_k]_+\Big) h^l \quad \text{modulo } \mathcal{S}^{\infty,2\delta} \\
\sim & Q[c]_+ \quad \text{modulo } \mathcal{S}^{\infty,2\delta}.
\end{aligned} \tag{5.88}$$

It follows that $[Qc]_- \sim Q[c]_-$ modulo $\mathcal{S}^{\infty,2\delta}$.

Let $q(x,\eta',\xi,z,h) \in S^0_h(1)$ be the (time independent) symbol of the pseudodifferential operator

$$Q = e^{-i\varphi_\infty/h} P(x,hD) e^{i\varphi_\infty/h}. \tag{5.89}$$

From (5.71), we get, for all $\varepsilon, N > 0$,

$$\big|\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma (hD_t + Q - z) a e^{i\varphi_\star/h}\big| \lesssim h^N e^{-(\operatorname{Re} S+\varepsilon)t}.$$

This estimate, combined with (5.75), gives

$$\big|\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma (hD_t + Q - z) b\big| \lesssim h^{-1-K_1-|\alpha|-|\beta|-|\gamma|} e^{-\delta t} + h^N e^{-(\operatorname{Re} S+\varepsilon)t}. \tag{5.90}$$

Since b is a classical expandible symbol, $d = \partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma (hD_t + Q - z) b$ is also a classical expandible symbol. Then (5.90) implies the

Lemma 5.12. *We have*

$$\big[\partial_t^\alpha \partial_x^\beta \partial_{\eta'}^\gamma (hD_t + Q - z) b\big]_- = 0 \quad \textit{modulo } \mathcal{S}^{\infty,2\delta}. \tag{5.91}$$

Proof. If there exists k such that $[d_k]_- \neq 0$, we set $\widehat{\jmath} < J_1$ the first index such that there exists k with $d_{k,\widehat{\jmath}} \neq 0$. Then, let $\widehat{k}$ be the first index with $d_{\widehat{k},\widehat{\jmath}} \neq 0$. Using (5.90), we get that, for all $N > 0$ and $\varepsilon > 0$,

$$\begin{aligned}
|d_{\widehat{k},\widehat{\jmath}}| \lesssim & h^{-C} e^{(\lambda_{\widehat{\jmath}} - 3\delta)t} + h^N e^{Ct} + \sum_{k<\widehat{k}} |d_k| h^{k-\widehat{k}} e^{\lambda_{\widehat{\jmath}} t} + h e^{\varepsilon t} \\
\lesssim & h^{-C} e^{(\lambda_{\widehat{\jmath}} - 3\delta)t} + h^N e^{Ct} + h^{-C} e^{(\lambda_{\widehat{\jmath}} - \lambda_{\widehat{\jmath}+1} + \varepsilon)t} + h e^{\varepsilon t},
\end{aligned} \tag{5.92}$$

where the constant C doesn't depend on $N, \varepsilon, t, h, x, \eta', z$. Notice that $\lambda_{\widehat{\jmath}} - 3\delta < 0$ and $\lambda_{\widehat{\jmath}} - \lambda_{\widehat{\jmath}+1} < 0$. Taking $h = e^{-\mu t}$ with $\mu > 0$ small enough, we get

$$|d_{\widehat{k},\widehat{\jmath}}| \lesssim e^{-\mu t/2}, \tag{5.93}$$

for ε small enough and N large enough. This implies $d_{\widehat{k},\widehat{\jmath}} = 0$, and this is a contradiction. $\square$

Now we finish the proof of Proposition 5.11. Using (5.71), (5.81), (5.88) and (5.91), we get

$$\begin{aligned}\big|\partial_t^\alpha\partial_x^\beta\partial_{\eta'}^\gamma(hD_t+&P(x,hD)-z)A_+\big|\\ &\lesssim h^N e^{(\operatorname{Re}S+\varepsilon)t}+\big|\partial_t^\alpha\partial_x^\beta\partial_{\eta'}^\gamma(hD_t+P(x,hD)-z)[b]_- e^{i\varphi_\infty/h}\big|\\ &=h^N e^{(\operatorname{Re}S+\varepsilon)t}+\big|\partial_t^\alpha\partial_x^\beta\partial_{\eta'}^\gamma(hD_t+Q(x,hD)-z)[b]_-\big|\\ &\lesssim h^N e^{(\operatorname{Re}S+\varepsilon)t}. \end{aligned}\tag{5.94}$$

The proposition follows, taking a geometric mean between the two estimates (5.75) and (5.94). □

Recalling that the functions

$$b_{k,j}(t,x,\eta',z)=\sum_l b_{k,j,l}(x,\eta',z)t^l,\tag{5.95}$$

are polynomial with respect to t, we can find a function A_-, holomorphic with respect to $z\in[-C_0h,C_0h]+i[-C_1h,C_1h]$, such that

$$A_-(x,\eta',z,h)\sim\sum_{k\geq 1-K_1}h^k\sum_{j<J_1,l}\frac{l!}{(S+\mu_j)^{l+1}}b_{k,j,l}(x,\eta',z).\tag{5.96}$$

Notice that, formally, $A_-=\int_0^{+\infty}\chi(t)[b]_-(t,x,\eta',z,h)dt$. At last, we set

$$u(x,\eta',z,h)=A_-(x,\eta',z,h)+\int_{-1}^{+\infty}\chi(t)A_+(t,x,\eta',z,h)dt,\tag{5.97}$$

and we have

Proposition 5.13 (see [14, Proposition 4.2])**.** *The function $u(x,\eta',z,h)$ is holomorphic with respect to $z\in[-C_0h,C_0h]+i[-C_1h,C_1h]$ and satisfies, for all $(\beta,\gamma)\in\mathbb{N}^{d+(d-1)}$,*

$$\partial_x^\beta\partial_{\eta'}^\gamma u=\mathcal{O}\big(h^{-K_1-|\beta|-|\gamma|}\big),\tag{5.98}$$

$$\partial_x^\beta\partial_{\eta'}^\gamma(P(x,hD)-z)u=\mathcal{O}(h^\infty),\tag{5.99}$$

for $x\in\bigcup_{t>-1/2}\widetilde{U}_t$ and $\eta'\in V_-$. Moreover, for $x\in H_-$, we have

$$u=(1+r(x,\eta',z,h))e^{ix'\cdot\eta'/h},\tag{5.100}$$

where $r\in\mathcal{S}^\infty(1)$.

Proof. The estimate (5.98) follows from (5.81) and (5.96). Now from (5.11), we get

$$\begin{aligned}\partial_x^\beta\partial_{\eta'}^\gamma(P(x,hD)-z)\int_{-1}^{+\infty}\chi A_+dt&=\int_{-1}^{+\infty}\partial_x^\beta\partial_{\eta'}^\gamma(hD_t+P(x,hD)-z)\chi A_+dt\\ &=\mathcal{O}(h^\infty)+\int_{-1}^{+\infty}\partial_x^\beta\partial_{\eta'}^\gamma(hD_t\chi)A_+dt\end{aligned}\tag{5.101}$$

so that the L.H.S. is microlocally 0 in Ω since $A_+=0$ microlocally in that set for $t\in\operatorname{supp}(\partial_t\chi)\subset]-1,-1/2[$.

On the other hand, from (5.91), we have

$$\sum_{1-K_1\leq\widetilde{k}\leq k} Q_{\widetilde{k}} b_{k-\widetilde{k},j,l} - \frac{z}{h} b_{k-1,j,l} = i(l+1)b_{k-1,j,l+1} - i(S+\mu_j)b_{k-1,j,l}. \tag{5.102}$$

Then

$$\begin{aligned} (P-z)A_- e^{i\varphi_\infty/h} =& e^{i\varphi_\infty/h}(Q-z)A_- \\ \sim& e^{i\varphi_\infty/h}\sum_k h^k \sum_{j<J_1,l} \frac{l!}{(S+\mu_j)^{l+1}}\Big(\sum_{1-K_1\leq\widetilde{k}\leq k} Q_{\widetilde{k}} b_{k-\widetilde{k},j,l} - \frac{z}{h} b_{k-1,j,l}\Big) \\ \sim& i e^{i\varphi_\infty/h}\sum_k h^k \Big(\sum_{j<J_1,l} \frac{(l+1)!}{(S+\mu_j)^{l+1}} b_{k-1,j,l+1} - \sum_{j<J_1,l} \frac{l!}{(S+\mu_j)^{l}} b_{k-1,j,l}\Big). \end{aligned} \tag{5.103}$$

Therefore $(P-z)A_- e^{i\varphi_\infty/h} = 0$ microlocally in Ω. One can also differentiate (5.103), and obtain the corresponding estimates. Then (5.99) follows from (5.101) and (5.103). Eventually, (5.100) follows from the fact that, for $x \in H_-$, a has a compact support in t: The formal stationary phase expansion (5.65) can be given a meaning, and gives this last estimate. □

6. The symbol of the transition operator

Now we finish the proof of Theorem 2.6. We compute the principal symbol of the operator $\mathcal{J}(z)$, defined after Theorem 2.5, that is the microlocal value of the solution u in Theorem 5.4 at some point $\rho_+ \in \Lambda_+ \setminus \widetilde{\Lambda_+}(\rho_-)$ (see the definition after (2.26)).

As in [14, Section 5], we can assume $K_1 = 0$ (see (5.73)) since the general case can be treated the same way. In that case, we recall that the solution u of the problem (5.5) can be written as

$$u(x,h) = \frac{1}{(2\pi h)^{d-\frac{1}{2}}} \iint_{T^*\mathbb{R}^{d-1}} \int_{-1}^{+\infty} e^{i(\varphi(t,x,\eta')-y'\eta')/h} a(t,x,\eta',z,h) u_0(\varepsilon,y') dt dy' d\eta', \tag{6.1}$$

where φ is defined in Section 5.1 and has the properties given in (5.57)–(5.58), and a is the symbol described in Section 5.2.

First of all, we compute the principal term a_0 of the symbol a in (6.1). Performing again a formal stationary phase with respect to t in (5.7), we obtain, for $x = (\varepsilon, x') \in H_-$,

$$u(x,h) = \frac{1}{(2\pi h)^{d-1}} \iint_{T^*\mathbb{R}^{d-1}} e^{i(x\cdot\eta'-y'\cdot\eta')/h} \widetilde{a}(x,\eta',z,h) u_0(\varepsilon,y') dy' d\eta', \tag{6.2}$$

where $\widetilde{a} = 1$ by our choice in (5.68). In particular for $x \in H_-$, we have

$$a_0(t(x,\eta'),x,\eta',z) = e^{-i\pi/4} |\partial^2_{tt}\varphi(t(x,\eta'),x,\eta')|^{1/2}. \tag{6.3}$$

Notice that we have done so that, microlocally near Λ_-,

$$\frac{1}{(2\pi h)^{d-1/2}} \int_{-1}^{+\infty} e^{i\varphi(t,x,\eta')/h} a(t,x,\eta',h,z) dt = b(x,\eta',h) e^{i\psi_{\eta'}(x)/h} \tag{6.4}$$

where $b(x,\eta',h)=(2\pi h)^{-(d-1)}\sum_{j=0}^{\infty}h^j b_j(x,\eta')$, is a symbol such that

$$\begin{cases} (P-z)\big(b(x,\eta',h)e^{i\psi_{\eta'}(x)/h}\big)=\mathcal{O}(h^\infty) & \text{near } \gamma_{\eta'},\\ b(x,\eta',h)=\tilde{a}(t(x,\eta'),x,\eta',z)=1 & \text{on } H_-, \end{cases} \tag{6.5}$$

The principal symbol b_0 of b satisfies

$$b_0(x,\eta')=e^{i\pi/4}\frac{a_0(t(x,\eta'),x,\eta',z,h)}{|\partial_{tt}^2\varphi(t(x,\eta'),x,\eta')|^{1/2}}, \tag{6.6}$$

and it is a solution of the first transport equation

$$\begin{cases} \partial_\xi p_0(x,\partial_x\psi_{\eta'})\partial_x b_0+\Big(\frac{1}{2}\operatorname{tr}\big(\partial_{\xi,\xi}^2 p_0(x,\partial_x\psi_{\eta'})\partial_{x,x}^2\psi_{\eta'}\big)-i\frac{z}{h}\Big)b_0=0 & \text{near } \gamma_{\eta'},\\ b_0(x,\eta')=\widetilde{a}_0(x',\eta')=1 & \text{on } H_-. \end{cases} \tag{6.7}$$

In the Schrödinger case, the first equation of (6.7) can be written as

$$2\nabla\psi_{\eta'}\cdot\nabla b_0+(\Delta\psi_{\eta'})b_0-\frac{iz}{h}b_0=0 \text{ near } \gamma_{\eta'}.$$

We calculate b_0, starting with the computation of the trace in (6.7) (as e.g. in the book of V. Maslov and M. Fedoryuk [19]). Let $(x(t,x',\eta'),\xi(t,x',\eta'))$ be the Hamiltonian curve with initial condition

$$(x(0,x',\eta'),\xi(0,x',\eta'))=(x_1^-,x',f_-(x_1^-,x',\eta'),\eta')\in\Lambda_{\psi_{\eta'}}. \tag{6.8}$$

With the notations of Lemma 5.6, this curve is $\gamma_{\eta'}$ when $x'=x'(\eta')$. As usual, we have

$$\partial_t\ln\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}=\operatorname{tr}\big(\partial_{\xi,\xi}^2 p_0(x,\partial_x\psi_{\eta'})\partial_{x,x}^2\psi_{\eta'}\big), \tag{6.9}$$

and then (6.7) becomes

$$\partial_t\left(\sqrt{\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}}b_0(x(t,x',\eta'),\eta')\right)=\frac{iz}{h}\sqrt{\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}}b_0(x(t,x',\eta'),\eta'), \tag{6.10}$$

which gives

$$b_0(x(t,x',\eta'),\eta')=\frac{\sqrt{\partial_{\xi_1}p_0(x_1^-,x',f_-(x_1^-,x',\eta'),\eta')}}{\sqrt{\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}}}e^{itz/h}. \tag{6.11}$$

We are interested in taking the limit $t\to+\infty$ in this expression. The point is that, as $t\to+\infty$,

$$\frac{1}{2}\partial_t\ln\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}=(\sum\lambda_j/2-\lambda_1)t+o(1). \tag{6.12}$$

Indeed, starting from (6.9), we have

$$\partial_{\xi,\xi}^2 p_0(x,\partial_x\psi_{\eta'})=2+\mathcal{O}(e^{-\lambda_1 t}), \tag{6.13}$$

as a matrix, for $x\in\gamma_{\eta'}(t)$. On the other hand, writing $\psi_{\eta'}(x)=\varphi(t(x,\eta'),x,\eta')$ and using the fact that $(\partial_t\varphi)(t(x,\eta'),x,\eta')=0$, we get

$$\partial_{x_j}\psi_{\eta'}=(\partial_{x_j}\varphi)(t(x,\eta'),x,\eta'), \tag{6.14}$$

so that

$$\partial^2_{x_j,x_k}\psi_{\eta'} = \partial^2_{x_j,x_k}\varphi + (\partial^2_{t,x_j}\varphi)(\partial_{x_k}t). \tag{6.15}$$

Now using (5.57), we have,

$$\partial^2_{x,x}\varphi = \partial^2_{x,x}\varphi_+ + \mathcal{O}(e^{-\lambda_1 t}) = \frac{L}{2} + \mathcal{O}(e^{-\lambda_1 t}), \tag{6.16}$$

and

$$\begin{aligned}\partial^2_{t,x_j}\varphi =& -\lambda_1(\partial_{x_j}\varphi_1)e^{-\lambda_1 t} + \mathcal{O}(e^{-\mu_2 t})\\ =& \begin{cases} \lambda_1^2|g_1|e^{-\lambda_1 t} + \mathcal{O}(e^{-\mu_2 t}) & \text{if } j=1,\\ \mathcal{O}(e^{-\mu_2 t}) & \text{if } j\neq 1.\end{cases}\end{aligned} \tag{6.17}$$

Then using again the fact that $(\partial_t\varphi)(t(x,\eta'),x,\eta')=0$, we get

$$\partial_{x_k}t = \begin{cases} -|g_1|^{-1}\lambda_1^{-1}e^{\lambda_1 t} + \mathcal{O}(e^{(\lambda_1-\widehat{\mu}_1)t}) & \text{if } k=1,\\ \mathcal{O}(e^{(\lambda_1-\widehat{\mu}_1)t}) & \text{if } k\neq 1.\end{cases} \tag{6.18}$$

Using (6.17) and the estimates (6.16), (6.17), (6.18), we obtain

$$\frac{1}{2}\operatorname{tr}\left(\partial^2_{\xi,\xi}p_0(x,\partial_x\psi_{\eta'})\partial^2_{x,x}\psi_{\eta'}\right) = \sum_{j=1}^{d}\lambda_j/2 - \lambda_1 + \mathcal{O}(e^{-\widehat{\mu}_1 t}), \tag{6.19}$$

on $\gamma_{\eta'}(t)$. Therefore, we shall write (6.11) as

$$\begin{aligned} b_0(x(t,x',\eta'),\eta') = e^{(\lambda_1 - S(z/h))t}&e^{(\sum\lambda_j/2-\lambda_1)t-\frac{1}{2}\ln\det\frac{\partial x(t,x',\eta')}{\partial(t,x')}}\\ &\times\sqrt{\partial_{\xi_1}p_0(x_1^-,x',f_-(x_1^-,x',\eta'),\eta')}. \end{aligned} \tag{6.20}$$

Now we compute $(\partial^2_{t,t}\varphi)(t(x,\eta'),x,\eta')$ on the curve $\gamma_{\eta'}$. We have $\partial_t\varphi = -p(x,\partial_x\varphi)$, and

$$\partial^2_{t,t}\varphi = -2(\partial_x\varphi)\cdot(\partial_{t,x}\varphi). \tag{6.21}$$

But, on $\gamma_{\eta'}$, we have (6.17) and

$$\begin{aligned}\partial_{x_j}\varphi =& \partial_{x_j}\varphi_+ + \partial_{x_j}\varphi_1 e^{-\lambda_1 t} + \mathcal{O}(e^{-\mu_2 t})\\ =& \begin{cases} -\dfrac{|g_1|\lambda_1}{2}e^{-\lambda_1 t} + \mathcal{O}(e^{-\mu_2 t}) & \text{if } j=1,\\ \mathcal{O}(e^{-\mu_2 t}) & \text{if } j\neq 1.\end{cases}\end{aligned} \tag{6.22}$$

Therefore, (6.21) becomes

$$\partial^2_{t,t}\varphi(t,x(t,x',\eta'),\eta') = |g_1|^2\lambda_1^3 e^{-2\lambda_1 t} + \mathcal{O}(e^{-(\lambda_1+\mu_2)t}). \tag{6.23}$$

We recall that a_0 is expandible, namely

$$a_0(t,x,\eta') \sim \sum_{j=0}^{\infty} a_{0,j}(t,x,\eta')e^{-(S(z/h)+\mu_j)t}, \tag{6.24}$$

where $a_{0,j}$ are polynomials with respect to t, and $a_{0,0}$ does not depend on t. Using (6.6), (6.20), (6.23) and (6.24), we get

$$\begin{aligned} a_{0,0}(x_{\eta'}(t),\eta') = |g_1|\lambda_1^{3/2}e^{-i\pi/4}&\sqrt{\partial_{\xi_1}p_0(x_1^-,x',f_-(x_1^-,x',\eta'),\eta')} \\ &\times e^{(\sum\lambda_k/2-\lambda_1)t-\frac{1}{2}\ln\det\frac{\partial x(t,x')}{\partial(t,x')}}b_0(x_{\eta'},\eta')+\mathcal{O}(e^{-\widehat{\mu}_1 t}), \end{aligned} \tag{6.25}$$

where $\widehat{\mu}_1=\mu_2-\mu_1$, and then, since $x_{\eta'}\in H_-$,

$$a_{0,0}(0,\eta') = |g_1|\lambda_1^{3/2}e^{-i\pi/4}\sqrt{\partial_{\xi_1}p_0(x_1^-,x',f_-(x_1^-,x',\eta'),\eta')}\lim_{t\to+\infty}\frac{e^{(\sum\lambda_k/2-\lambda_1)t}}{\sqrt{\det\frac{\partial x(t,x')}{\partial(t,x')}}}. \tag{6.26}$$

Notice that the above limit exists thanks to (6.19).

Finally we compute the solution $u(x,h)$ given by (5.7) microlocally near ρ_+. Since $\rho_+\in\Lambda_+\setminus\widetilde{\Lambda_+}(\rho_-)$, we can use the calculus of [14, Section 5] and we get, microlocally near ρ_+,

$$\int_{-1}^{+\infty}e^{i\varphi(t,x,\eta')/h}a(t,x,\eta',z,h)dt = c(x,\eta',h)e^{i(\varphi_+(x)+\tilde{\psi}(\eta'))/h}. \tag{6.27}$$

Here $c(x,\eta',h)$ is a symbol of class $\mathcal{S}_h^0$ which satisfies

$$c(x,\eta',h)\sim\sum_{j=0}^{\infty}c_j(x,\eta',\ln h)h^{S(z/h)/\lambda_1+\widehat{\mu}_j/\lambda_1}, \tag{6.28}$$

where the $c_j(x,\eta',\ln h)$ are polynomial with respect to $\ln h$ and, in particular,

$$c_0(x,\eta')=\frac{1}{\lambda_1}(\varphi_1(x)/i)^{-S(z/h)/\lambda_1}\Gamma\left(S(z/h)/\lambda_1\right)a_{0,0}(x,\eta'), \tag{6.29}$$

doesn't depend on $\ln h$. Here Γ denotes Euler's Gamma function, and $(\widehat{\mu}_j)_{j\geq 0}$ is the increasing sequence of the linear combinations over $\mathbb{N}$ of the $(\mu_k-\mu_1)$'s, $k\geq 2$.

On the other hand, since we want that the function $u(x,h)$, given by

$$u(x,h)=\frac{1}{(2\pi h)^{d-1/2}}\iint_{T^*\mathbb{R}^{d-1}}c(x,\eta',h)e^{i(\varphi_+(x)+\tilde{\psi}(\eta')-y'\cdot\eta')/h}u_0(y)dy'd\eta', \tag{6.30}$$

is a microlocal solution of $(P-z)u=0$ for any initial data u_0, the function c_0 should satisfy the usual transport equation:

$$\partial_\xi p_0(x,\partial_x\varphi_+)\partial_x c_0+\Big(\frac{1}{2}\operatorname{tr}\left(\partial^2_{\xi,\xi}p_0(x,\partial_x\varphi_+)\partial^2_{x,x}\varphi_+\right)-i\frac{z}{h}\Big)c_0=0. \tag{6.31}$$

Thus, if $(x(t),\xi(t))$ is the integral curve of H_p in Λ_+ with initial condition $\rho=(x,\nabla\varphi_+(x))$, we have

$$c_0(x(t),\eta')=e^{itz/h-\frac{1}{2}\int_0^t\operatorname{tr}(\partial^2_{\xi,\xi}p_0(\cdot,\partial_x\varphi_+)\partial^2_{x,x}\varphi_+)(x(s))ds}c_0(x,\eta'). \tag{6.32}$$

Let us compute $c_0(x(t),\eta')$ using (6.29). Since $\rho_+\notin\widetilde{\Lambda_+}(\rho_-)$, we can assume that $\rho\notin\widetilde{\Lambda_+}(\rho_-)$ for ρ close enough to ρ_+. In particular $\rho\notin\widetilde{\Lambda_+}$ and then

$$x(t)\sim\sum_{j=1}^{\infty}g_j^+(t)e^{\mu_j t}, \tag{6.33}$$

as $t \to -\infty$, where the $g_j^+(t)$ are polynomials with respect to t and $g_1^+(t)$ doesn't depend on t.

$$\varphi_1(x(t)) = -\lambda_1(g_1^-(\rho_{\eta'}) \cdot g_1^+(\rho))e^{\lambda_1 t} + \mathcal{O}(e^{(\mu_2-\varepsilon)t}), \tag{6.34}$$

and the equations (6.29) and (6.32) give

$$\begin{aligned} c_0(x,\eta') =& \frac{1}{\lambda_1} e^{\frac{1}{2}\int_0^t (\operatorname{tr}(\partial^2_{\xi,\xi} p_0(\cdot,\partial_x\varphi_+)\partial^2_{x,x}\varphi_+)(x(s)) - \sum \lambda_l)ds} (i\lambda_1(g_1^-(\rho_{\eta'}) \cdot g_1^+(\rho)))^{-S(z/h)/\lambda_1} \\ & \Gamma\left(S(z/h)/\lambda_1\right) a_{0,0}(x(t),\eta') + \mathcal{O}(e^{\varepsilon t}) \\ =& \frac{1}{\lambda_1} e^{\frac{1}{2}\int_0^{-\infty} (\operatorname{tr}(\partial^2_{\xi,\xi} p_0(\cdot,\partial_x\varphi_+)\partial^2_{x,x}\varphi_+)(x(s)) - \sum \lambda_j)ds} (i\lambda_1(g_1^-(\rho_{\eta'}) \cdot g_1^+(\rho)))^{-S(z/h)/\lambda_1} \\ & \Gamma\left(S(z/h)/\lambda_1\right) a_{0,0}(0,\eta'). \end{aligned} \tag{6.35}$$

At last, we go back to (6.30) and we perform a stationary phase expansion with respect to η' in that integral. Recalling (5.58), we can write

$$u(x,h) = \frac{1}{(2\pi h)^{d-1/2}} \iint_{T^*\mathbb{R}^{d-1}} e^{i\varphi(x,\eta',y')/h} c(x,\eta',h) u_0(y) dy' d\eta', \tag{6.36}$$

where

$$\varphi(x,\eta',y') = \varphi_+(x) + (x'(\eta') - y') \cdot \eta' - \varphi_-(x(\eta')). \tag{6.37}$$

We have

$$\nabla_{\eta'}\varphi(x,\eta',y') = (x'(\eta') - y') + \nabla_{\eta'} x'(\eta') \cdot (\eta' - \nabla_{x'}\varphi_-(x(\eta'))), \tag{6.38}$$

since $x(\eta') = (x_1^-, x'(\eta'))$ where x_1^- does not depend on η'. But $\rho(\eta') = (x(\eta'), \xi(\eta'))$ belongs to Λ_- (see (5.20)), so that $\nabla\varphi_-(x(\eta')) = \xi(\eta')$, and in particular $\nabla_{x'}\varphi_-(x(\eta')) = \eta'$. Thus the last term in (6.38) vanishes, and $\eta' \mapsto \varphi(x,\eta',y')$ has a unique critical point $\eta'(y')$, such that $y' = x'(\eta'(y'))$, with critical value

$$\tilde{\varphi}(x,y') = \varphi(x,\eta'(y'),y') = \varphi_+(x) - \varphi_-(\varepsilon,y'). \tag{6.39}$$

Moreover, since $\nabla^2_{\eta' x'}\varphi_-(x(\eta')) = I$, we have

$$\nabla^2_{\eta'\eta'}\varphi(x,\eta',y') = \nabla_{\eta'} x'(\eta') = \left(\nabla^2_{x'x'}\varphi_-(x(\eta'))\right)^{-1}. \tag{6.40}$$

Thus, there exists a symbol $d(x,y',z,h) \sim \sum_{j\geq 0} d_j(x,y',z,\ln h) h^{\widehat{\mu}_j/\lambda_1} \in \mathcal{S}_h^0(1)$, with $d_j(x,y',z,\ln h)$ polynomial with respect to $\ln h$, such that

$$\mathcal{J}(z)u_0(x,h) = \frac{h^{S(z/h)/\lambda_1}}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} d(x,y',z,h) e^{i(\varphi_+(x) - \varphi_-(\varepsilon,y'))/h} u_0(\varepsilon,y') dy', \tag{6.41}$$

microlocally near ρ_+. Moreover the principal symbol d_0 of d is independent of $\ln h$, and can be written as

$$d_0(x,y',z) = e^{-i(d-1)\pi/4} |\det \nabla^2_{y'y'}\varphi_-(\varepsilon,y')|^{1/2} c_0(x,\eta'(y')), \tag{6.42}$$

where $c_0(x,\eta'(y'))$ is given by (6.35) and (6.26), and this finishes the proof Theorem 2.6.

Appendix A. A review of h-pseudodifferential calculus

One of the main tool of this paper is the so-called h-pseudodifferential calculus, and we review here some basic facts. Since we deal with self-adjoint operators and spectral properties, we shall only use Weyl quantization. First we recall this calculus in standard classes of symbols, following closely [9, Chapter 7] (see also [17]).

We say that $m : T^*\mathbb{R}^d \to [0, +\infty[$ is an order function when there are C, $N > 0$ such that $m(x) \leq C\langle x - y\rangle^N m(y)$.

If $m(x, \xi)$ is an order function, and $\delta \geq 0$ a real number, we say that a function $a(x, \xi, h) \in C^\infty(T^*\mathbb{R}^d)$ is a symbol of class $\mathcal{S}_h^\delta(m)$ when

$$\forall \alpha \in \mathbb{N}^{2d},\ \exists C_\alpha > 0,\ \forall h \in]0, 1],\ |\partial_{x,\xi}^\alpha a(x, \xi, h)| \leq C_\alpha h^{-\delta|\alpha|} m(x, \xi). \tag{A.1}$$

If $e(h)$ is a function of h only, sometimes we write $\mathcal{S}_h^\delta(e(h)m)$ instead of $e(h)\mathcal{S}_h^\delta(m)$.

If $a(x, \xi, h)$ is a symbol of class $\mathcal{S}_h^\delta(m)$, we define the h-pseudodifferential operator $\mathrm{Op}_h(a)$ with symbol a by

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d),\ (\mathrm{Op}_h(a)u)(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi. \tag{A.2}$$

We also denote by $\Psi_h^\delta(m)$ the space of operators $\mathrm{Op}_h(\mathcal{S}_h^\delta(m))$.

The composition rule between pseudodifferential operators in $\Psi_h^\delta(m)$ is given in the following proposition. It is an easy adaptation of Proposition 7.7 in [9]:

Proposition A.1. *If $a_1 \in \mathcal{S}_h^{\delta_1}(m_1)$ and $a_2 \in \mathcal{S}_h^{\delta_2}(m_2)$ with $0 \leq \delta_1, \delta_2 \leq \frac{1}{2}$ and $\delta_1 + \delta_2 < 1$, then $\mathrm{Op}_h(a_1) \circ \mathrm{Op}_h(a_2)$ belongs to $\Psi_h^{\max(\delta_1,\delta_2)}(m_1 m_2)$, and, for any $N \in \mathbb{N}$, its symbol $a_1 \# a_2$ verifies*

$$\begin{aligned}
(a_1 \# a_2)(x, \xi) &= e^{\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)}\big(a_1(x, \xi) a_2(y, \eta)\big)\Big|_{y=x,\eta=\xi} \\
&= \sum_{k=0}^{N-1} \frac{1}{k!}\Big(\big(\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\big)^k a_1(x, \xi) a_2(y, \eta)\Big)\Big|_{y=x,\eta=\xi} \\
&\quad + h^{N(1-\delta_1-\delta_2)} \mathcal{S}_h^{\max(\delta_1,\delta_2)}(m_1 m_2).
\end{aligned} \tag{A.3}$$

Notice that in this theorem and below, we use the standard notation $\sigma(D_x, D_\xi, D_y, D_\eta) = D_\xi D_y - D_x D_\eta$.

To control the norm of a pseudodifferential operator in $\mathcal{L}(L^2(\mathbb{R}^d))$, we use the following classical result:

Theorem A.2. *(Calderòn–Vaillancourt) Let $a \in \mathcal{S}_h^\delta(1)$ with $0 \leq \delta \leq 1/2$. Then there exists $C > 0$ such that*

$$\forall u \in L^2(\mathbb{R}^d),\ \|\mathrm{Op}_h(a)u\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{L^2(\mathbb{R}^d)}. \tag{A.4}$$

Furthermore, C is bounded by a semi-norm of $a \in \mathcal{S}_h^\delta(1)$.

We now recall the semiclassical sharp Gårding inequality and Fefferman–Phong's inequality:

Theorem A.3. *(Gårding's inequality) Let $a(x,\xi,h)$ be a real valued symbol in $\mathcal{S}_h^0(1)$. If $a(x,\xi,h)\geq 0$ for all $(x,\xi,h)\in T^*\mathbb{R}^d\times[0,1]$, then there exists $C>0$ such that*

$$\forall u\in L^2(\mathbb{R}^d),\ \big(\operatorname{Op}_h(a)u,u\big)_{L^2(\mathbb{R}^d)}\geq -Ch\|u\|^2_{L^2(\mathbb{R}^d)}. \tag{A.5}$$

Furthermore, C is bounded by a semi-norm of $a\in\mathcal{S}_h^0(1)$.

Theorem A.4. *(Fefferman-Phong's inequality) Let $a(x,\xi,h)$ be a real valued symbol in $\mathcal{S}_h^0(1)$. If $a(x,\xi,h)\geq 0$ for all $(x,\xi,h)\in T^*\mathbb{R}^d\times[0,1]$, then there exists $C>0$ such that*

$$\forall u\in L^2(\mathbb{R}^d),\ \big(\operatorname{Op}_h(a)u,u\big)_{L^2(\mathbb{R}^d)}\geq -Ch^2\|u\|^2_{L^2(\mathbb{R}^d)}. \tag{A.6}$$

Furthermore, C is bounded by a semi-norm of $a\in\mathcal{S}_h^0(1)$.

We now give the composition rule in the class $\widetilde{\mathcal{S}}_h$ we use in Section 4, which can be seen as a particular case of the semiclassical Weyl–Hörmander calculus. Let $m(x,\xi)$ be an order function. We say that a function $a(x,\xi,h)$ is a symbol of class $\widetilde{\mathcal{S}}_h(m)$ if $\forall\alpha,\beta\in\mathbb{N}^d$, $\exists C_{\alpha,\beta}>0$ such that , $\forall h\ll 1$,

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi,h)|\leq C_{\alpha,\beta}m(x,\xi)\langle x\rangle^{-|\alpha|/2}\langle\xi\rangle^{-|\beta|/2}. \tag{A.7}$$

Concerning the product rule, we have the following result, which is similar to Proposition A.1:

Proposition A.5. *If $a_1\in\widetilde{\mathcal{S}}_h(m_1)$ and $a_2\in\widetilde{\mathcal{S}}_h(m_2)$, then $\operatorname{Op}_h(a_1)\circ\operatorname{Op}_h(a_2)$ is a pseudodifferential operator of class $\widetilde{\mathcal{S}}_h(m_1m_2)$ and its symbol is given by*

$$a\#b(x,\xi) = e^{\frac{ih}{2}\sigma(D_x,D_\xi,D_y,D_\eta)}\big(a(x,\xi)b(y,\eta)\big)\Big|_{y=x,\eta=\xi} \tag{A.8}$$

$$= \sum_{k=0}^{N-1}\frac{1}{k!}\Big(\big(\frac{ih}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\big)^k a(x,\xi)b(y,\eta)\Big)\Big|_{y=x,\eta=\xi} \tag{A.9}$$

$$+ h^N\widetilde{\mathcal{S}}_h\big(\langle x\rangle^{-N/2}\langle\xi\rangle^{-N/2}m_1m_2\big). \tag{A.10}$$

Proof. We follow the proof of [9, Proposition 7.7]. Since $a_j\in\mathcal{S}_h^0(m_j)$, we now that $\operatorname{Op}_h(a_1)\circ\operatorname{Op}_h(a_2)$ is a pseudodifferential operator whose symbol in $\mathcal{S}_h^0(m_1m_2)$ is given by (A.8). Let $X=(x,y,\xi,\eta)$, $\widetilde{X}=(\widetilde{x},\widetilde{y},\widetilde{\xi},\widetilde{\eta})$ and $\chi\in C_0^\infty(\mathbb{R}^{4d})$ be equal to 1 near 0. Using Fourier's inversion formula, one can show that, if we set

$$I(X)=e^{\frac{ih}{2}\sigma(D_{\widetilde{X}})}\Big(\chi\Big(\frac{X-\widetilde{X}}{\langle X\rangle^\mu}\Big)a_1a_2(\widetilde{X})\Big)(X)-\sum_{j\leq N-1}\frac{1}{j!}\left(\frac{ih}{2}\sigma(D_X)\right)^j a_1a_2(X), \tag{A.11}$$

we have

$$|I(X)|\lesssim h^N\sum_{|\alpha|\leq d/2+1}\Big\|D^\alpha_{\widetilde{X}}\sigma(D_{\widetilde{X}})^N\Big(\chi\Big(\frac{X-\widetilde{X}}{\langle X\rangle^\mu}\Big)a_1a_2(\widetilde{X})\Big)\Big\|_{L^2_{\widetilde{X}}}. \tag{A.12}$$

Now, using the estimate (A.7), we have

$$\begin{aligned}
&\Big| D^\alpha_{\widetilde{X}} \sigma(D_{\widetilde{X}})^N \Big(\chi\Big(\frac{X-\widetilde{X}}{\langle X\rangle^\mu}\Big) a_1a_2(\widetilde{X})\Big)\Big| \\
&\qquad\lesssim \sum_{|\beta|+|\gamma|=N} \Big| D^\alpha_{\widetilde{X}} \partial^\beta_{\widetilde{x}} \partial^\beta_{\widetilde{\eta}} \partial^\gamma_{\widetilde{y}} \partial^\gamma_{\widetilde{\xi}} \Big(\chi\Big(\frac{X-\widetilde{X}}{\langle X\rangle^\mu}\Big) a_1a_2(\widetilde{X})\Big)\Big| \\
&\qquad\lesssim \sum_{|\beta|+|\gamma|=N} \sum_{\substack{0\leq j,k\leq|\beta| \\ 0\leq l,m\leq|\gamma|}} \langle X\rangle^{-j\mu} \langle \widetilde{x}\rangle^{-(|\beta|-j)/2} \langle X\rangle^{-k\mu} \langle \widetilde{\eta}\rangle^{-(|\beta|-k)/2} \\
&\qquad\qquad \langle X\rangle^{-l\mu} \langle \widetilde{y}\rangle^{-(|\gamma|-l)/2} \langle X\rangle^{-m\mu} \langle \widetilde{\xi}\rangle^{-(|\gamma|-m)/2} m_1(\widetilde{X}) m_2(\widetilde{X}).
\end{aligned} \tag{A.13}$$

But since $\widetilde{X}$ is in the support of $\chi\big((X-\widetilde{X})\langle X\rangle^{-\mu}\big)$, we also have

$$\langle X\rangle^{-j\mu} \langle \widetilde{x}\rangle^{-(|\beta|-j)/2} \lesssim \begin{cases} \langle X\rangle^{-2|\beta|\mu/3} & \text{for } j\geq 2|\beta|/3 \\ \langle x\rangle^{-|\beta|/6} \langle X\rangle^{|\beta|\mu/2} & \text{for } j\leq 2|\beta|/3 \end{cases} \tag{A.14}$$

Therefore, using the fact that m_j are order functions, we obtain the estimate

$$\begin{aligned}
|I(X)| \lesssim & h^N \langle X\rangle^{N_0} m_1m_2(X) \sum_{|\beta|+|\gamma|=N} \big(\langle X\rangle^{-2|\beta|\mu/3} + \langle x\rangle^{-|\beta|/6} \langle X\rangle^{|\beta|\mu/2}\big) \\
& \big(\langle X\rangle^{-2|\beta|\mu/3} + \langle \eta\rangle^{-|\beta|/6} \langle X\rangle^{|\beta|\mu/2}\big) \big(\langle X\rangle^{-2|\gamma|\mu/3} + \langle y\rangle^{-|\gamma|/6} \langle X\rangle^{|\beta|\mu/2}\big) \\
& \big(\langle X\rangle^{-2|\gamma|\mu/3} + \langle \xi\rangle^{-|\gamma|/3} \langle X\rangle^{|\beta|\mu/6}\big),
\end{aligned} \tag{A.15}$$

where N_0 is independent of N.

Now, if we assume $y=x$ and $\eta=\xi$, we have,

$$\begin{aligned}
\big(\langle X\rangle^{-2|\beta|\mu/3} + \langle x\rangle^{-|\beta|/6} \langle X\rangle^{|\beta|\mu/2}\big) & \big(\langle X\rangle^{-2|\beta|\mu/3} + \langle \eta\rangle^{-|\beta|/6} \langle X\rangle^{|\beta|\mu/2}\big) \\
& \lesssim \langle x,\xi\rangle^{-|\beta|\mu/6} + \langle x\rangle^{-|\beta|/6} \langle \xi\rangle^{-|\beta|/6} \langle x,\xi\rangle^{|\beta|\mu} \\
& \lesssim \langle x,\xi\rangle^{-|\beta|(1/6-\mu)},
\end{aligned} \tag{A.16}$$

so that (A.15) gives

$$|I(x,\xi,x,\xi)| \lesssim h^N \langle x,\xi\rangle^{-(1/6-\mu)N+N_0} m_1(x,\xi) m_2(x,\xi). \tag{A.17}$$

One obtains the same way the same estimate for the derivatives of I, and we are left with the estimate of

$$J(X) := e^{\frac{ih}{2}\sigma(D_{\widetilde{X}})} \Big(\Big(1-\chi\Big(\frac{X-\widetilde{X}}{\langle X\rangle^\mu}\Big)\Big) a_1a_2(\widetilde{X})\Big)(X) \tag{A.18}$$

$$= \frac{1}{(2\pi h)^{2d}} \int e^{-2i\sigma(X-\widetilde{X})/h} \big(1-\chi\big((X-\widetilde{X})\langle X\rangle^{-\mu}\big)\big) a_1a_2(\widetilde{X}) d\widetilde{X}. \tag{A.19}$$

We make integrations by parts, using the operator

$${}^tL = \big(\partial_{\widetilde{X}} \sigma(X-\widetilde{X})\big)^{-2} \big(\partial_{\widetilde{X}} \sigma(X-\widetilde{X})\big).\partial_{\widetilde{X}}. \tag{A.20}$$

At each integration, we gain a factor h and an $|X-\widetilde{X}|^{-1}$, which is lower than $\langle X\rangle^{-\mu}$ on the support of $1-\chi$. Then, for each $M \gg 1$,

$$\begin{aligned} |J(X)| \lesssim & h^{M-2d}\langle X\rangle^{-(M-M_0)\mu} \sum_{|\alpha|\leq M} \big\| \langle X-\widetilde{X}\rangle^{M_0} \partial_{\widetilde{X}}^{\alpha} a_1 a_2 \big\|_{L^1_{\widetilde{X}}} \\ \lesssim & h^{M-2d}\langle X\rangle^{-(M-M_0)\mu} m_1(X) m_2(X) \big\| \langle X-\widetilde{X}\rangle^{M_0} \langle X-\widetilde{X}\rangle^{M_0} \big\|_{L^1_{\widetilde{X}}}. \end{aligned} \tag{A.21}$$

Eventually, if $y=x$ and $\eta=\xi$, we get, for all $M\in\mathbb{N}$,

$$|J(x,\xi,x,\xi)| \lesssim h^M \langle x,\xi\rangle^{-M} m_1(x,\xi) m_2(x,\xi). \tag{A.22}$$

We can prove also the same estimates for the derivatives of J, and the proposition follows from (A.17) and (A.22). □

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Jean-François Bony, Laboratoire MAB, UMR CNRS 5466, Université de Bordeaux I, Bordeaux, France

E-mail address: `jean-francois.bony@math.u-bordeaux1.fr`

Setsuro Fujiie, Graduate School of Material Science, University of Hyogo, Japan

E-mail address: `fujiie@sci.u-hyogo.ac.jp`

Thierry Ramond, Mathématiques, Université Paris XI, UMR CNRS 8628, Orsay, France

E-mail address: `thierry.ramond@math.u-psud.fr`

Maher Zerzeri, Département de Mathématiques, UMR CNRS 7539, Institut Galilée, Université Paris XIII, Villetaneuse, France

E-mail address: `zerzeri@math.univ-paris13.fr`